CHAPTER 1

DPP 1.1

Single Correct Answer Type

1. (c) $0 < x < \frac{\pi}{2}$

$$\therefore \quad [x] = \begin{cases} 0 & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad 1 \le x < \frac{\pi}{2} \end{cases}$$

$$\Rightarrow \quad \sin[x] = \begin{cases} \sin 0 = 0 & \text{if} \quad 0 < x < 1 \\ \sin 1 & \text{if} \quad 1 \le x < \frac{\pi}{2} \end{cases}$$

We have $0 < \sin x < 1$ when $0 < x < \frac{\pi}{2}$.

$$\therefore \quad [\sin x] = 0 \text{ for } 0 < x < \frac{\pi}{2}$$

$$\therefore \quad \sin[x] + [\sin x] = \begin{cases} 0 & \text{if} \quad 0 < x < 1 \\ \sin 1 & \text{if} \quad 1 \le x < \frac{\pi}{2} \end{cases}$$

2. (b) Here $\phi(-x) = \frac{1}{1+e^x}$.

So, $\quad \phi(x) + \phi(-x) = \frac{1}{1+e^{-x}} + \frac{1}{1+e^x}$

$$= \frac{e^x}{e^x+1} + \frac{1}{1+e^x} = \frac{e^x+1}{e^x+1} = 1$$

$$\therefore \quad S = \{\phi(5) + \phi(-5)\} + \ldots + \{\phi(1) + \phi(-1)\} + \phi(0)$$

$$= 1 + 1 + 1 + 1 + 1 + \phi(0) = 5 + \frac{1}{1+e^0} = 5 + \frac{1}{2} = \frac{11}{2}$$

3. (c) $f(1, 1) = f(0, f(1, 0)) = f(0, f(0, 1)) = f(0, 2) = 3$

4. (b) $f: R \to Q$ and $g: R \to Q$, but $\sqrt{3} f(x) + g(x) = 4$ it is true only, when $\quad f(x) = 0 \Rightarrow g(x) = 4$

$\therefore \quad (1 - f(x))^3 + (g(x) - 3)^3 = 2$

5. (c) $g(n) = f(n) + f(n+1)$

If n is even, $n+1$ is odd.

$\therefore \quad g(n) = 2n - 1 + 2(n+1) + 2 = 4n + 3$

If n is odd, $n+1$ is even.

$\therefore \quad g(n) = 2n + 2 + 2(n+1) - 1 = 4n + 3.$

6. (c) $\log_{|x^2-1|} |x-3|$ is defined when $x - 3 \neq 0$ and $|x^2 - 1| > 0$ and $|x^2 - 1| \neq 1$.

$\Rightarrow \quad x \neq 3;$

$x^2 - 1 \neq 0 \quad$ i.e., $\quad x \neq 1$ or -1;

$|x^2 - 1| \neq 1 \quad \Rightarrow x \neq -\sqrt{2}$ or $\sqrt{2}$; and $x \neq 0$

Hence, the points at which the function is not defined are $x = 0, 1, -1, \sqrt{2}, -\sqrt{2}, 3$.

7. (a) Here, $\log \frac{y^2}{x(y-1)} = 0$

or $\quad \frac{y^2}{x(y-1)} = 1$

or $\quad y^2 = xy - x$

or $\quad y^2 - xy + x = 0$

$\therefore \quad y = \frac{x \pm \sqrt{x^2 - 4x}}{2}$

So y is real if

$x^2 - 4x \ge 0$

or $\quad x(x-4) \ge 0$

$\therefore \quad x \le 0$ or $x \ge 4$.

But $x > 0$ for $\log x$ to be defined.

So, $\quad x \ge 4$

Now, $\quad x = \frac{y^2}{y-1} \ge 4$

or $\quad \frac{y^2 - 4y + 4}{y-1} \ge 0$

or $\quad (y-2)^2/(y-1) \ge 0$

$\Rightarrow \quad y \ge 1$

But $y > 1$ for $\log (y-1)$ to be real.

8. (d) We have,

	$y = [x^2] - [x]^2,$	$x \in [0, 2]$
i.e.,	$y = [x^2],$	$0 \le x < 1$
	$y = [x^2] - 1,$	$1 \le x < 2$
	$y = 0$	$x = 2$
i.e.,	$y = 0,$	$0 \le x < 1$
	$= 1 - 1 = 0,$	$1 \le x < \sqrt{2}$
	$= 2 - 1 = 1,$	$\sqrt{2} \le x < \sqrt{3}$
	$= 3 - 1 = 2,$	$\sqrt{3} \le x < 2$
	$= 0$	$x = 2$

Hence, the range is $\{0, 1, 2\}$.

9. (d)

$$f(x) = \sin^{-1}\left(\frac{x^2 - 2x}{3}\right) + \sqrt{[x] + [-x]}$$

$\sin^{-1}\left(\frac{x^2-2x}{3}\right)$ is defined for $-1 \le \frac{x^2-2x}{3} \le 1$

and $\sqrt{[x] - [-x]}$ is defined only for integral values of x

$\Rightarrow \quad x = -1, 0, 1, 2, 3$

10. (d) We have $f(x) = \sqrt{-\sin^{-1}(\sqrt{\log_4 x})} + \sin^{-1}\left(\frac{1+x^2}{4x}\right)$

Clearly, domain of $f(x)$ is $x = 1$ only, so $f(1) = \sqrt{0} + \sin^{-1}\left(\frac{2}{4}\right) = \frac{\pi}{6}$

Hence, range of $f(x)$ is $\left\{\frac{\pi}{6}\right\}$.

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11. (c) We have

$$f(x) = \tan^{-1}\left(\frac{x^2+1}{x^2+\sqrt{3}}\right), x \in R$$

Let $y = \frac{x^2+1}{x^2+\sqrt{3}}$

$\Rightarrow \quad yx^2 + \sqrt{3}\,y = x^2 + 1$

$\Rightarrow \quad (\sqrt{3}\,y - 1) = (1 - y)\,x^2$

$\therefore \quad x^2 = \left(\frac{\sqrt{3}y-1}{1-y}\right) \geq 0$

So, $\frac{\sqrt{3}y-1}{y-1} \leq 0$

$\Rightarrow \quad y \in \left[\frac{1}{\sqrt{3}}, 1\right)$

$\therefore$ Range of $f(x) = \left[\frac{\pi}{6}, \frac{\pi}{4}\right)$

12. (c) We must have $x^4 - 21x^2 \geq 0$ and $10 - \sqrt{x^4 - 21x^2} \geq 0$

$\Rightarrow \quad x^2(x^2 - 21) \geq 0$ (1)

and $100 \geq x^4 - 21x^2$ (2)

Eq. (1) gives $x = 0$ or $x \leq -\sqrt{21}$ or $x \geq \sqrt{21}$

Eq. (2) $\Rightarrow x^4 - 21x^2 - 100 \leq 0$

$\Rightarrow \quad (x^2 - 25)(x^2 + 4) \leq 0$

$\Rightarrow \quad x^2 - 25 \leq 0$ (as $x^2 + 4 > 0$ always)

$\Rightarrow \quad -5 \leq x \leq 5$

Domain is given by $[-5, -\sqrt{21}] \cup [\sqrt{21}, 5]$ and $x = 0$.

13. (c) $f(x) = \log_{[x^2]}(4 - |x|) + \log_2\{\sqrt{x}\}$

For $4 - |x| > 0 \Rightarrow |x| < 4 \Rightarrow -4 < x < 4$

For $[x^2] > 0$, $[x^2] \neq 1$

$\Rightarrow [x^2] \geq 2$

$\Rightarrow 2 \leq x^2 < \infty$

$\Rightarrow \sqrt{2} \leq x < \infty$ or $-\infty < x \leq -\sqrt{2}$

For $\{\sqrt{x}\} > 0$, $\sqrt{x} \neq i \Rightarrow x \neq i^2$

$\therefore \quad x = 2, 3$

14. (a) We have $f(x) = \log_e\{\text{sgn}(9 - x^2)\} + \sqrt{[x]^3 - 4[x]}$

We must have, $\text{sgn}(9 - x^2) > 0$

$\Rightarrow \quad 9 - x^2 > 0$

$\Rightarrow \quad x^2 - 9 < 0$

$\Rightarrow \quad (x - 3)(x + 3) < 0$ (i)

$\Rightarrow \quad -3 < x < 3$

Also $[x]^3 - 4[x] \geq 0$

$\Rightarrow \quad [x]([x]^2 - 4) \geq 0$

$\Rightarrow \quad [x]([x] - 2)([x] + 2) \geq 0$

$\Rightarrow \quad [x] \geq 2$ or $[x]$ lies between -2 and 0,

i.e. $[x] = -2, -1$ or 0

Now, $[x] \geq 2 \Rightarrow x \geq 2$ (ii)

$[x] = -2 \Rightarrow -2 \leq x < 1$

$[x] = -1 \Rightarrow -1 \leq x < 0$

$[x] = 0 \Rightarrow 0 \leq x < 1$.

Hence $[x] = -2, -1, 0 \Rightarrow -2 \leq x < 1$

Hence $D_f = [-2, 1) \cup [2, 3)$.

15. (c) Here, $f(x) = \frac{1}{\pi}(\sin^{-1} x + \tan^{-1} x) + \frac{1}{(x+1) + \frac{4}{(x+1)}}$

$= g(x) + h(x)$

where, domain of $g(x) \in [-1, 1]$

$\therefore$ Maximum value of $g(x) = g(1) = \frac{3}{4}$ and minimum value of $g(x) = g(-1) = -\frac{3}{4}$

Also, maximum value of $h(x)$ occurs when $(x + 1) + \frac{4}{(x+1)}$ is minimum at $x = 1$.

Therefore, range of $f(x)$ is $\left[-\frac{3}{4}, 1\right]$.

Multiple Correct Answers Type

16. (a, b, c)

$1 \leq |\sin x| + |\cos x| \leq \sqrt{2}$

$\therefore \quad 2[|\sin x| + |\cos x|] = 2$

$\therefore f(x)$ is defined if

$\sin^2 x + 2 \sin x + \frac{11}{4} \geq 2$

or $(\sin x + 1)^2 \geq \frac{1}{4}$

or $\sin x + 1 \geq \frac{1}{2}$ or $\sin x + 1 \leq -\frac{1}{2}$

or $\sin x \geq -\frac{1}{2}$ or $\sin \leq -\frac{3}{2}$ (which is not true)

17. (a, b, d)

(a) Clearly, $\sin[x] + \cos[x]$ is defined for all x.

(b) $\sin x$ is always defined and $1 + \sin^2 x \geq 1$

$\Rightarrow \sec^{-1}(1 + \sin^2 x)$ is defined for all x, as $1 + \sin^2 x \geq 1$

(c) $\tan(\log x)$ is not defined if $\log x = (2k + 1)\frac{\pi}{2}$

(d) $\frac{9}{8} + \cos x + \cos 2x$

$= 2\cos^2 x + \cos x + \frac{1}{8} > 0, \forall x \in R$ as $D < 0$

Hence, domain is R.

18. (a, c)

$f(x) = \frac{2x}{1+x^2}$, $[-2, 2]$ has range $[-1, 1]$, hence bounded.

$f(x) = \frac{x^2}{1-x}, x \in [0, 2] - \{1\}$ is unbounded as $f(x) \to \infty$, when $x \to 1^-$

$f(x) = \frac{x^3 - 8x + 6}{4x + 1}$, $[0, 5]$ is bounded because domain does not contain $-1/4$.

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DPP 1.2

Single Correct Answer Type

1. (b) $G(x)=\left(\frac{1}{a^x-1}+\frac{1}{2}\right)F(x)$

$\therefore\ G(-x)=\left(\frac{1}{a^{-x}-1}+\frac{1}{2}\right)F(-x)$

$=-\left(\frac{a^x}{1-a^x}+\frac{1}{2}\right)F(x)$

$=\left(\frac{a^x}{a^x-1}-\frac{1}{2}\right)F(x)$

$=\left(\frac{a^x-1+1}{a^x-1}-\frac{1}{2}\right)F(x)$

$=\left(1+\frac{1}{a^x-1}-\frac{1}{2}\right)F(x)$

$=\left(\frac{1}{a^x-1}+\frac{1}{2}\right)F(x)=G(x)$

$\therefore\ G(x)$ is an even function.

2. (d) $f(x)=\alpha x^3-\beta x-(\tan x)\ \text{sgn}\ x$

Since $f(-x)=f(x)$

$\Rightarrow\quad -\alpha x^3+\beta x-\tan x\ \text{sgn}\ x=\alpha x^3-\beta x-(\tan x)\ (\text{sgn}\ x)$

$\Rightarrow\quad 2(\alpha x^2-\beta)x=0\ \forall\ x\in R$

$\Rightarrow\quad \alpha=0$ and $\beta=0$

$\therefore\quad [a]^2-5[a]+4=0$ and $6\{a\}^2-5\{a\}+1=0$

$\Rightarrow\quad ([a]-1)([a]-4)=0$ and $(3\{x\}-1)(2\{x\}-1)=0$

$\Rightarrow\quad [a]=1, 4$ and $\{a\}=1/3, 1/2$

$\therefore\quad a=1+\frac{1}{3}, 1+\frac{1}{2}, 4+\frac{1}{3}, 4+\frac{1}{2}$

Sum of values of $a=\frac{35}{3}$

3. (c) $f(x)=\sin^2x+(1-\sin^2x)^2+2$

$=3-\sin^2x+\sin^4x$

$=3-\sin^2x\cos^2x$

$=3-\frac{\sin^2 2x}{4}$

$\Rightarrow\quad T_1=\frac{\pi}{2}$, and $T_2=\frac{\pi}{2}$

7. (b) $f(x)=\frac{\sin x+\sin 3x}{\cos x+\cos 3x}$

$=\frac{2\sin 2x\cos x}{2\cos 2x\cos x}=\tan 2x$

where $x\neq(2n+1)\frac{\pi}{4}, (2n+1)\frac{\pi}{2}$

$f\left(x+\frac{\pi}{2}\right)=\tan 2\left(x+\frac{\pi}{2}\right)=\tan(2x+\pi)=\tan 2x$

it seems that $\frac{\pi}{2}$ is a period but it is not because $f(0)$ is defined where as $f\left(\frac{\pi}{2}\right)$ is not defined.

Thus, period is π.

8. (b) $f(x+1)+f(x+3)=2$ (i)

Replacing x by $x+2$, we get

$f(x+3)+f(x+5)=2$ (ii)

Subtracting (ii) rom (i), we get

$f(x+1)=f(x+5)$

or $f(x)=f(x+4)$

9. (a) Let $f(x)=\text{sgn}([x]+[-x])$

$=\begin{cases}0; & x\in I\\ -1; & x\notin I\end{cases}$

Hence $f(x)$ is periodic with period 1.

10. (c) $f(x)=f(-x)$ where $f(x)=x|x|+\sin|x|+xe^x$,

$\therefore\quad F(x)=x^2-\sin x-xe^{-x}$ (i)

Also, $g(x)=-g(-x)$, where $g(x)=\cos x+x^2-x$

$\therefore\quad G(x)=-(\cos x+x^2+x)=-\cos x-x^2-x$ (ii)

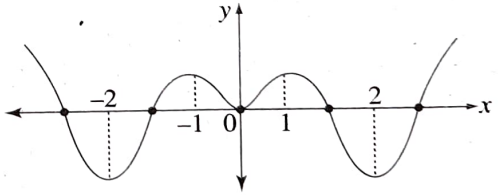
$\therefore\quad F(x)+G(x)=-\sin x-xe^{-x}-\cos x-x$

$=-(\sin x+\cos x+x+xe^{-x})$

11. (a) $P(x)$ is an even function.

So, it is symmetrical about y-axis.

$P(-1)=P(1)=1$ and $P(-2)=P(2)=-5$



Graph of $P(x)$ showing minimum number of distinct real zeroes

Multiple Correct Answers Type

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Also, $g(x) = \begin{cases} 0, & x \in I \\ g(-x), & x \notin I \end{cases}$

Thus $g(x)$ is periodic function

13. (b, c, d)

We have $f(x) = f(2 - x)$

Replacing x by $f(1 + x)$, we get

$f(1 + x) = f(1 - x)$

Hence graph of $f(x)$ is symmetrical about the line $x = 1$.

Also $g(x) = f(1 + x) = f(1 - x) = g(-x)$

$\therefore$ $g(x)$ is an even function.

Further $f'(1 + x) = -f'(1 - x)$

$\therefore$ $f'(1) = -f'(1)$

$\therefore$ $f'(1) = 0$

DPP 1.3

Single Correct Answer Type

1. (a) If $f(x)$ is surjective then range of $f(x)$ must be $[1, \infty)$.

$\therefore$ Range of $\sqrt{3x^2 - 4x + k + 1} + 10 \in [10, \infty)$

$\Rightarrow$ Range of $3x^2 - 4x + k + 1 \in [0, \infty)$

$\Rightarrow$ $D = 0$

$\Rightarrow$ $16 - 12(k + 1) = 0$

$\Rightarrow$ $4 - 3k - 3 = 0$

$\Rightarrow$ $k = \frac{1}{3}$

2. (b) $f(x) = (1 + \sqrt{-x}) - (\sqrt{-x} - x)$

$= 1 - x, x \le 0$

Then $1 - x \ge 1$.

Hence function is injective and surjective.

3. (a) $x, y \in A$ and $f(x) = f(y) \Rightarrow f(f(x)) = f(f(y))$. So, f is one-one

Also $a \in A \Rightarrow f(f(a)) = a \Rightarrow f(b) = a$ where $b = f(a) \in A$. So, f is onto.

4. (a) Clearly $f(x) \to \infty$ when $x \to \infty$

and $f(x) \to -\infty$ when $x \to -\infty$

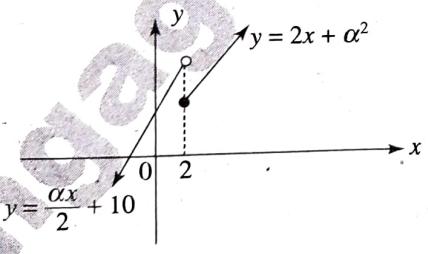
$\therefore$ Range is R as $f(x)$ is continuous.

Also for $x \ge 0, f'(x) = |x| - \sin x \ge 0$

$\therefore$ $f(x)$ is increasing

$\therefore$ $f(x)$ is one-one.

5. (c)



Clearly from graph

we must have

$2(2) + \alpha^2 \le \alpha + 10$

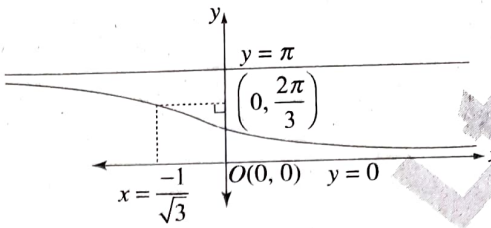
$\therefore$ $(\alpha - 3)(\alpha + 2) \le 0$

$\therefore$ $\alpha \in [-2, 3]$

Also $\alpha > 0$

$\therefore$ $\alpha \in (0, 3]$

6. (b)



Graph of $y = \cot^{-1} x$

Clearly

$\cot^{-1}(x^2 - 4x + \alpha) > \frac{2\pi}{3}, \forall x \in R.$

$\therefore$ $x^2 - 4x + \alpha > \frac{-1}{\sqrt{3}}, \forall x \in R$

$\Rightarrow$ $x^2 - 4x + \alpha + \frac{1}{\sqrt{3}} > 0 \ \forall \ x \in R$

So, discriminant $< 0 \Rightarrow 16 - 4\left(\alpha + \frac{1}{\sqrt{3}}\right) < 0$

$4 - \alpha - \frac{1}{\sqrt{3}} < 0 \Rightarrow \alpha > 4 - \frac{1}{\sqrt{3}}$

$\therefore$ $\alpha \in \left(4 - \frac{1}{\sqrt{3}}, \infty\right)$

Hence, minimum integral α is 4.

7. (b) $f(x) = \text{sgn}\,(x - x^4 + x^7 - x^8 - 1)$

For $x \in (0, 1); x - 1 < 0, x^7 - x^4 < 0$

$\therefore$ $x - x^4 + x^7 - x^8 - 1 < 0$

For $x \in (1, \infty); x < x^4, x^7 < x^8$

$\therefore$ $x - x^4 + x^7 - x^8 - 1 < 0$

Also for $x = 1; x - x^4 + x^7 - x^8 - 1 = -1$

Thus $x - x^4 + x^7 - x^8 - 1 < 0$ for all $x \in R^+$

Hence sgn $(x - x^4 + x^7 - x^8 - 1) = -1 \ \forall \ x \in R^+$

Therefore $f(x)$ is many-one and into

8. (d) I. $f(x) = x$ and $g(x) = -x$ or $f(x) = x$ and $g(x) = -x^3$

II. $f(x) = x$ and $g(x) = x^3$

III. $f(x) = \sin x$ which is odd but not one-one or $f(x) = x^2 \sin x$ which is odd but many one

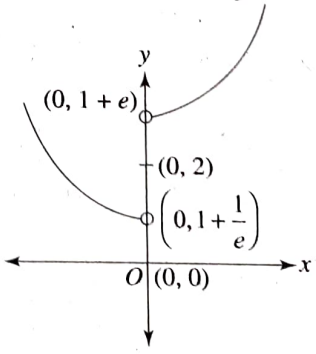
9. (c)

(a) $f(x) = e^{\text{sgn}\,x} + e^{x^2}$

when $x = 0$ then $f(0) = 2$

when $x > 0$ then $f(x) = e + e^{x^2}$

when $x < 0$ then $f(x) = \frac{1}{e} + e^{x^2}$



Hence, $f(x)$ is many-one

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(b) We have $f(x) = e^{x^2+|x|}, x \in [-1, \infty)$

clearly $f(-1) = e^2 = f(1)$

also $x^2 + |x| \geq 0 \ \forall \ x \in [-1, \infty]$

$\Rightarrow$ $R_f = [1, \infty)$

$\therefore$ $f(x)$ is many-one into function.

(c) $f(x) = |x-1| + |x-2| + |x-3| + |x-4|, x \in [3, 4]$

$\therefore$ $f(x) = (x-1) + (x-2) + (x-3) - (x-4)$

$= (3x-6) - (x-4)$

$f(x) = 2x - 2$, which is increasing function

$R_f = [4, 6]$

Clearly, $f(x)$ is one-one onto function.

(d) $f(x) = \sqrt{\ln(\cos(\sin x))}$

For domain, $\ln(\cos(\sin x)) \geq 0$

$\Rightarrow$ $(\cos(\sin x)) \geq 1$

$\Rightarrow$ $\cos(\sin x) = 1$;

$\therefore$ $\sin x = 0$

$\Rightarrow$ $x = n\pi, n \in I$

$R_f = \{0\}$

since $f(x) = 0$

Thus, $f(x)$ is many-one function.

Comprehension Type

10. (c), 11. (a), 12.(a)

$f(x) = (x-1)^2 - 2$

$\Rightarrow$ $a = 1, b = -2$

Now, $g(x) = x^2 + x - 2 = \left(x + \frac{1}{2}\right)^2 - \frac{9}{4}$

$\Rightarrow$ $g(|x|) = \left(|x| + \frac{1}{2}\right)^2 - \frac{9}{4}$

$\therefore$ $g_{\min} = g(0) = -2$

Now, $f: [1, \infty) \to [-2, \infty)$,

$\Rightarrow$ $f^{-1}: [-2, \infty) \to [1, \infty)$;

$f(x) = y$

$\Rightarrow$ $x^2 - 2x - (1 + y) = 0$

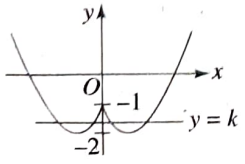
$\Rightarrow$ $x = \frac{2 \pm \sqrt{4 + 4(1+y)}}{2}$

$\Rightarrow$ $x = 1 \pm \sqrt{2+y}$

So, $f^{-1}(y) = 1 + \sqrt{2+y}$;

$f^{-1}(x) = 1 + \sqrt{2+x}$

Graph of $y = f(|x|)$



From graph $f(|x|) = k$ has four roots if $k \in (-2, -1)$.

DPP 1.4

Single Correct Answer Type

1. (d) We have $4\cos^4 x - 2\cos 2x - \frac{1}{2}\cos 4x - x^7$

$= 4\cos^4 x - 2(2\cos^2 x - 1) - \frac{1}{2}(2\cos^2 2x - 1) - x^7$

$= 4\cos^4 x - 4\cos^2 x + 2 - (2\cos^2 x - 1)^2 + \frac{1}{2} - x^7$

$= \left(\frac{3}{2} - x^7\right)$

$\Rightarrow$ $g(x) = \left(\frac{3}{2} - x^7\right)^{\frac{1}{7}}$

$\Rightarrow$ $g(g(x)) = \left(\frac{3}{2} - (g(x))^7\right)^{\frac{1}{7}} = \left(\frac{3}{2} - \left(\frac{3}{2} - x^7\right)\right)^{\frac{1}{7}} = x$

Hence $g(g(100)) = 100$

2. (c) $f(x) = x^2 + x + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{1}{2} \geq \frac{1}{2}$

$g(f(x)) = (f(x))^2 + af(x) + 1$

for $g(f(x)) = 0$,

$a = -\left(f(x) + \frac{1}{f(x)}\right) \leq -2$

$\therefore$ If $a > -2$, $g(f(x)) = 0$ has no solutions

3. (a) $1 \leq \log_2 (x^2 + 3x - 2) \leq 3$

$\Rightarrow$ $2 \leq (x^2 + 3x - 2) \leq 8$

$\Rightarrow$ $-5 \leq x \leq -4$ and $1 \leq x \leq 2$

4. (b) $f(g(x)) = \frac{rx}{1 + (r-1)x}$, $g(f(x)) = rx$.

If $f(g(x)) = g(f(x))$

$\Rightarrow$ $\frac{rx}{1 + (r-1)x} = rx$

$\Rightarrow$ $rx\left[1 - \frac{1}{1 + (r-1)x}\right] = 0$

If this is to be true for infinitely many (all) x, then $r = 0$ or $r - 1 = 0$

5. (a) $fof(x) = \frac{a\left[\frac{ax+b}{cx+d}\right] + b}{c\left[\frac{ax+b}{cx+d}\right] + d} = x$

$\therefore$ $(ac + dc)x^2 + (bc + d^2 - bc - a^2)x - ab - bd = 0$

It is true for all real x,

$\therefore$ $ac + dc = 0, a^2 - d^2 = 0, ab + bd = 0$

so $a = -d$

6. (a) $f^{-1}og^{-1}(23) = (gof)^{-1}(23)$

Now $gof(x) = 2(x^3 + 3) + 1$

For $2(x^3 + 3) + 1 = 23$

$\Rightarrow$ $x = 2$

$\therefore$ $gof^{-1}(23) = 2$

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7. (c) $\{x \in R : f^{-1}(x) = f(x)\}$

$\equiv \{x \in R : f(f(x)) = x\}$

$f(f(x)) = f(x(x-1)) = [x(x-1)][x(x-1)-1]$

$= x(x-1)[x^2 - x - 1]$

Now, $f(f(x)) = x$

$\Rightarrow \quad x(x-1)(x^2 - x - 1) = x$

$\Rightarrow \quad x(x^3 - 2x^2) = 0$

$\Rightarrow \quad x = 2 \ (\because x \in [\frac{1}{2}, \infty))$

8. (a) $y = 2\log_a x$

$\Rightarrow \quad \log_a x = \frac{y}{2}$

$\Rightarrow \quad x = a^{y/2}$

$\Rightarrow \quad f^{-1}(y) = a^{y/2}$

$\Rightarrow \quad f^{-1}(b+c) = a^{\frac{b+c}{2}} = a^{\frac{b}{2}} \cdot a^{\frac{c}{2}} = f^{-1}(b) \cdot f^{-1}(c)$

9. (c) $f(g(x)) = \begin{cases} x^2 + 1 & -1 \le x \le 1 \\ 2x^2 + 1 & 1 < x \le \sqrt{2} \end{cases}$

$f(g(x)) = 2 \Rightarrow x^2 + 1 = 2$ and $2x^2 + 1 = 2$

$\Rightarrow \quad x = \pm 1, x = \pm\frac{1}{\sqrt{2}}$

No. of solutions = 2

10. (d) $f(g(x)) = a(bx + a) + b$

$= abx + a^2 + b$ (i)

$g(f(x)) = b(ax + b) + a$

$= abx + b^2 + a$ (ii)

From (i) – (ii), we get

$f(g(20)) - g(f(20)) = a^2 - b^2 + b - a$

$\therefore \quad (a^2 - b^2) + (b - a) = 28$

$\therefore \quad (a-b)(a+b-1) = 28 = 1 \times 28$ or 2×14 or 4×7

If $a - b = 1$ and $a + b - 1 = 28$

Then $a = 15;\ b = 14$

If $a - b = 2$ and $a + b - 1 = 14$ (not possible)

If $a - b = 4$ and $a + b - 1 = 7$

Then $a = 6$ and $b = 2$

11. (d) Given $g(x) = 2f(x) + 5$

Replacing x by $g^{-1}(x)$, we get

$g(g^{-1}(x)) = 2f(g^{-1}(x)) + 5$

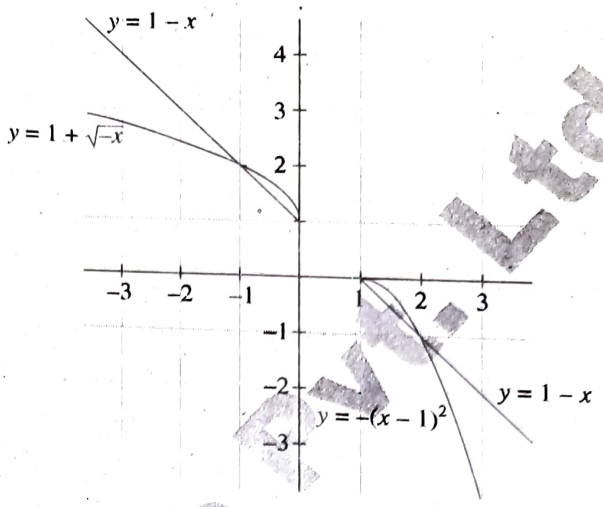
$\Rightarrow \quad x = 2f(g^{-1}(x)) + 5$

$\Rightarrow \quad f(g^{-1}(x)) = \frac{x-5}{2}$

$\Rightarrow \quad g^{-1}(x) = f^{-1}\left(\frac{x-5}{2}\right)$

12. (d) $f^{-1}(x) = \begin{cases} -x + 1, & x \ge 1 \\ 1 + \sqrt{-x}, & x \le 0 \end{cases}$ the solutions of

$f(x) - f^{-1}(x) = 0$ are $x = -1, 1, 0, 2$.



DPP 1.5

Single Correct Answer Type

1. (b) Let $f\left(x + \frac{1}{y}\right) + f\left(x - \frac{1}{y}\right) = 2f(x)f\left(\frac{1}{y}\right) \forall x, y \in R$

Given $f(0) = 0$

Putting $x = 0, y = \frac{1}{x}$, we get

$f(x) + f(-x) = 2f(0)f(x)$

$\Rightarrow \quad f(x) + f(-x) = 0$

$\Rightarrow \quad f(x) = -f(-x)$

Putting $x = 1, y = -1$, we get

$\Rightarrow \quad f(2) + f(0) = 2[f(1)]^2$

$\Rightarrow \quad f(2) = 2[f(1)]^2$

Putting $x = -1, y = -1$, we get

$\Rightarrow \quad f(-2) = 2f(-1)f(-1)$

$\Rightarrow \quad -f(2) = 2(f(1))^2$

$\Rightarrow \quad f(2) = -f(2)$

$\Rightarrow \quad f(2) = 0$

$\Rightarrow \quad f(1) = 0$

$\therefore \quad f(1) = f(2) = 0$

$\therefore \quad f(1) + f(2) = 0$

2. (c) $x^2 - 6x + 6 = x^2 - 4x + 4 \Rightarrow x = 1$

Put $x = 1$ in the given relation to get

$f(1) = 1$

for $x^2 - 6x + 6 = 1$, $x = 1$ or 5

for $x^2 - 4x + 4 = 1$, $x = 1$ or 3

Put $x = 3$ and 5 in the given relation to get

$f(-3) + f(1) = 6 \Rightarrow f(-3) = 5$

and $f(1) + f(9) = 10 \Rightarrow f(9) = 9$

$\therefore \quad f(-3) + f(9) - 5f(1) = 9$

3. (c) We have,

$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$

Put $x = f(y) = 0$

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Then $f(0) = f(0) + 0 + f(0) - 1$

$\therefore \quad f(0) = 1$

Putting $f(y) = x$

$\therefore \quad f(0) = f(x) + x^2 + f(x) - 1$

Hence, $f(x) = 1 - \frac{x^2}{2}$

4. (a) Let $f(0) = k$. Then $f(x) = f(x + 0) = f(0) = k$

So, f is a constant function. But $f\left(\frac{3}{4}\right) = \frac{3}{4}$

$\therefore \quad f(x) = \left(\frac{3}{4}\right)$ for all x and hence $f\left(\frac{9}{16}\right) = \frac{3}{4}$

5. (c) Taking $x = y = 1$, we get

$f(1)f(1) - f(1) = 2$

$\Rightarrow \quad f^2(1) - f(1) - 2 = 0$

$\Rightarrow \quad (f(1) - 2)(f(1) + 1) = 0$

$\Rightarrow \quad f(1) = 2 \qquad (\text{as } f(1) > 0)$

Taking $y = 1$, we get

$f(x) \cdot f(1) - f(x) = x + 1$

$\Rightarrow \quad f(x) = x + 1$

$\Rightarrow \quad f^{-1}(x) = x - 1$

$\therefore \quad f(x) \cdot f^{-1}(x) = x^2 - 1$

6. (b) $(f(xy))^2 = x(f(y))^2$ and $f(2) = 6$

Put $x = 25$ and $y = 2$

$(f(50))^2 = 25(f(2))^2 = 25 \times 36$

$\therefore \quad f(50) = 30$

7. (b) $f(x + f(x)) = 4f(x)$ and $f(1) = 4$

Put $x = 1$, then

$f(1 + f(1)) = 4f(1)$

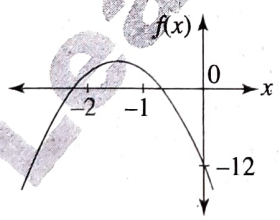
$\Rightarrow \quad f(5) = 16$

Put $x = 5$, then

$f(5 + f(5)) = 4f(5)$

$\Rightarrow \quad f(21) = 64$

8. (c) $g(x) \in [-2, -1) \ \forall \ x \in R$



$f(x) = -3x^2 - kx - 12$

$\Rightarrow \quad f(0) = -12$

$\because \quad f(g(x)) > 0 \ \forall \ x \in R$

$\therefore \quad f(-2) > 0$ and $f(-1) \geq 0$

$f(-2) > 0 \Rightarrow -12 + 2k - 12 > 0 \Rightarrow k > 12$

$f(-1) \geq 0 \Rightarrow -3 + k - 12 \geq 0 \Rightarrow k \geq 15$

$\Rightarrow k \geq 15$

9. (c) Given that $f(f(n)) = n$ and (i)

$f(0) = 1$

Put $n = 0$, we get

$f(f(0)) = 0$ or $f(1) = 0$

Also given that $f(f(n + 2) + 2) = n$ (ii)

Put $n = -1$, we get

$f(f(1) + 2) = -1$ or $f(0 + 2) = -1$ or $f(2) = -1$

For $f(3)$, put $n = -2$ in (ii)

$\therefore \quad f(f(0) + 2) = -2$

or $\quad f(1 + 2) = -2$

or $\quad f(3) = -2$

Multiple Correct Answers Type

10. (a, b, c)

$(x - y)f(x + y) - (x + y)f(x - y) = 2y(x - y)(x + y)$

Let $x - y = u$; $x + y = v$. Then

$uf(v) - vf(u) = uv(v - u)$

$\Rightarrow \quad \frac{f(v)}{v} - \frac{f(u)}{u} = v - u$

$\Rightarrow \quad \left(\frac{f(v)}{v} - v\right) = \left(\frac{f(u)}{u} - u\right) = \text{constant}$

Let $\frac{f(x)}{x} - x = \lambda$. Then

$f(x) = (\lambda x + x^2)$

Since $f(1) = 2$ then $\lambda = 1$

$\therefore \quad f(x) = x^2 + x$

11. (a, c)

$f(x + y) - kxy = f(x) + 2y^2$ (1)

Put $y = -x$

$\Rightarrow \quad f(0) + kx^2 = f(x) + 2x^2$ (2)

$\Rightarrow \quad f(x) = (k - 2)x^2 + f(0)$

Put $x = 1$

$\therefore \quad f(1) = (k - 2) + f(0)$

$\therefore \quad k - 2 + f(0) = 2$

$\therefore \quad k + f(0) = 4$ (3)

Put $x = 1$

$\therefore \quad f(2) = (k - 2) \cdot 4 + f(0)$

$\therefore \quad 4k + f(0) = 16$

Solving we get $3k = 12$ or $k = 4$ and $f(0) = 0$

$\therefore \quad f(x) = 2x^2$

12. (a, b)

$f(1000)\, f(f(1000)) = 1$

$\Rightarrow \quad f(1000)\, f(999) = 1$

$\Rightarrow \quad 999\, f(999) = 1$

$\therefore \quad f(999) = \frac{1}{999}$

The numbers 999 and $\frac{1}{999}$ are in the range of f.

Hence, by intermediate value property of continuous function, function takes all values between 999 and $\frac{1}{999}$, then there exists

$\alpha \in \left(\frac{1}{999}, 999\right)$ such that $f(\alpha) = 500$

Then $f(\alpha)f(f(\alpha)) = 1 \Rightarrow f(500) = \frac{1}{500}$

Similarly, $199 \in \left(\frac{1}{199}, 999\right)$, thus $f(199) = \frac{1}{199}$

But there is nothing to show that 1999 lies in the range of f.

Thus (d) is not correct and (c) is also incorrect.

13. (a, b, c)

(a) $-1 \leq |x| - 1 \leq 4 \quad \Rightarrow 0 \leq |x| \leq 5 \quad \Rightarrow -5 \leq x \leq 5$

(b) $f(|x| + 1)$

range of $f(|x| + 1)$ is $[0, 2]$

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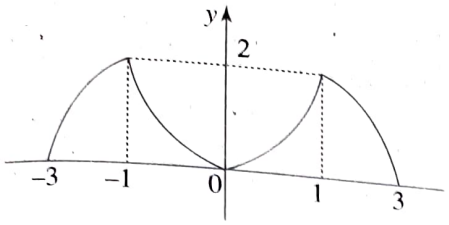
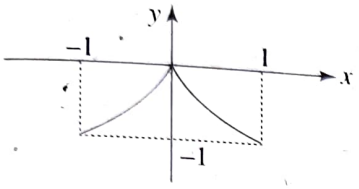


(c) range of $f(-|x|)$ is $[-1, 0]$



(d) $-1 \le |x| \le 4 \Rightarrow x \in [-4, 4]$

14. (a, b, c, d)

Comprehension Type

(b), 16. (c)

$$f(x+y) = 2^x f(y) + 4^y f(x) \quad \text{(i)}$$

Interchanging x and y, we get

$$f(x+y) = 2^y f(x) + 4^x f(y) \quad \text{(ii)}$$

$$\Rightarrow \quad 2^x f(y) + 4^y f(x) = 2^y f(x) + 4^x f(y)$$

$$\Rightarrow \quad \frac{f(x)}{4^x - 2^x} = \frac{f(y)}{4^y - 2^y} = k$$

$$\Rightarrow \quad f(x) = k(4^x - 2^x)$$

$$\Rightarrow \quad f(1) = k(4 - 2) = 2$$

$$\Rightarrow \quad k = 1.$$

Hence, $f(x) = 4^x - 2^x$.

$$f(4) = 4^4 - 2^4 = 240$$

$$f(x) = (2^x)^2 - 2^x = (2^x - 1/2)^2 - 1/4$$

Thus, $f(x)$ has least value as $-1/4$.

17. (b), 18. (a)

Putting $x = y = 0$, we get

$$f(0) = \frac{f(0) + f(0)}{1 - [f(0)]^2}$$

$$\Rightarrow \quad f(0)[f^2(0) + 1] = 0 \Rightarrow f(0) = 0 \text{ (since } f^2(0) \neq -1).$$

Now putting $y = -x$, we get

$$f(0) = \frac{f(x) + f(-x)}{1 - f(x) \cdot f(-x)}$$

$$\Rightarrow \quad f(x) + f(-x) = 0$$

$$\Rightarrow \quad f(-x) = -f(x) \Rightarrow f(x) \text{ is an odd function.}$$

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CHAPTER 2

DPP 2.1

Single Correct Answer Type

1. (c) Given that,

$$f(x) = [x-3] + |x-4|$$

$$\therefore \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} ([x-3] + |x-4|)$$

$$= \lim_{h\to 0} ([3-h-3] + |3-h-4|)$$

$$= \lim_{h\to 0} ([-h] + 1 + h)$$

$$= -1 + 1 + 0 = 0$$

2. (a) $\lim\limits_{x\to \frac{\pi}{2}^+} \left[\dfrac{[\sin x] - [\cos x] + 1}{3}\right] = \left[\dfrac{0-(-1)+1}{3}\right] = 0$

$\lim\limits_{x\to \frac{\pi}{2}^-} \left[\dfrac{[\sin x] - [\cos x] + 1}{3}\right] = \left[\dfrac{0-0+1}{3}\right] = 0$

$$\therefore \lim_{x\to \frac{\pi}{2}} \left[\frac{[\sin x] - [\cos x] + 1}{3}\right] = 0$$

3. (c) $x < -\dfrac{1}{3}$

$$\Rightarrow \frac{1}{x} > -3 \Rightarrow -\frac{1}{x} < 3 \Rightarrow \left[-\frac{1}{x}\right] = 2, \text{ when } x \to \frac{-1^-}{3}$$

$$\Rightarrow \lim_{x\to \frac{-1^-}{3}} \frac{1}{x}\left[\frac{-1}{x}\right] = (-3)(2) = -6$$

4. (c) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \left(x + \dfrac{1}{2}\right) = \dfrac{1}{2}$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \left(2x + \dfrac{3}{4}\right) = \dfrac{3}{4}$

$\therefore \lim\limits_{x\to 0} f(x)$ does not exist.

5. (a) $L = \lim\limits_{x\to -7} \dfrac{[x]^2 + 15[x] + 56}{\sin(x+7)\sin(x+8)}$

$$= \lim_{x\to -7} \frac{([x]+7)([x]+8)}{\sin(x+7)\sin(x+8)}$$

Now, $\lim\limits_{x\to -7^+} \dfrac{([x]+7)([x]+8)}{\sin(x+7)\sin(x+8)} = \dfrac{(\text{exact } 0)\cdot(1)}{(\to 0)\sin(-1)} = 0$

$\lim\limits_{x\to -7^-} \dfrac{([x]+7)([x]+8)}{\sin(x+7)\sin(x+8)} = \dfrac{(-1)\cdot(\text{exact } 0)}{(\to 0)\sin(-1)} = 0$

$\therefore L = 0$

6. (c)

(a) $\lim\limits_{x\to 0^+} (x)^{\log_e x} = (0^+)^{-\infty} = \infty$

So, limit does not exist finitely.

(b) $\lim\limits_{x\to 3} \dfrac{x^2 - 9 - \sqrt{x^2 - 6x + 9}}{|x-1| - 2}$

$$= \lim_{x\to 3} \frac{x^2 - 9 - |x-3|}{x-3}$$

Now,

$$f(3^+) = \lim_{x\to 3^+} \frac{(x^2-9)-(x-3)}{(x-3)} = \lim_{x\to 3^+} ((x+3)-1) = 5$$

Also,

$$f(3^-) = \lim_{x\to 3^-} \frac{(x^2-9)+(x-3)}{(x-3)} = \lim_{x\to 3^-} ((x+3)+1) = 7$$

So, limit does not exist.

(c) $\lim\limits_{x\to 1^+} ([x])^{\frac{1}{x-1}} = \lim\limits_{x\to 1^+} \dfrac{1}{x-1} = 1$

7. (d) If (variable)$^{\text{variable}}$ and base is negative then limit does not exist.

And $\lim\limits_{x\to 4} (x-6)^4 = 16$.

8. (c) We must have

$$-1 \le \frac{4\tan^{-1}(2\pi x)}{\pi} \le 1$$

$$\Rightarrow -\frac{\pi}{4} \le \tan^{-1} 2\pi x \le \frac{\pi}{4}$$

$$\Rightarrow -1 \le 2\pi x \le 1$$

$$\Rightarrow x \in \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$$

9. (c) $\lim\limits_{x\to\infty} \left[x - \log_e\left(\dfrac{e^x + e^{-x}}{2}\right)\right]$

$$= \lim_{x\to\infty} \left[x - \log_e e^x\left(\frac{1+e^{-2x}}{2}\right)\right]$$

$$= \lim_{x\to\infty} \left[x - x - \log_e\left(\frac{1+e^{-2x}}{2}\right)\right]$$

$$= -\log_e\left(\frac{1}{2}\right) = \log_e 2$$

10. (b) $\lim\limits_{x\to\infty} (e^x + \pi^x)^{1/x} = \pi \lim\limits_{x\to\infty} \left(\left(\dfrac{e}{\pi}\right)^x + 1\right)^{1/x} = \pi$

Now, $\{\pi\} = \pi - 3$

$$\therefore \lim_{x\to\infty} \left\{(e^x + \pi^x)^{\frac{1}{x}}\right\} = \pi - 3$$

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11. (c) If $\frac{\cos x}{\sin ax}$ is periodic then a must be rational.

Now, a is rational and $n \to \infty$. Thus, $n!\,\pi a$ will become an integral multiple of π.

$\therefore \quad \lim_{n\to\infty} (1+\cos^{2m} n!\pi a) = 1 + (\pm 1)^{2m} = 2$

Multiple Correct Answers Type

12. (b, c)

$$A = \lim_{x\to 0} \frac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)}$$

$$\lim_{x\to 0^-} \frac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)} = \lim_{x\to 0^+} \frac{x}{-x} = -1$$

$$\lim_{x\to 0^+} \frac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)} = \lim_{x\to 0} \frac{x}{x} = 1$$

Thus, A does not exist.

$$B = \lim_{x\to 0} \frac{[|x|]}{x} = 0 \text{ as } [|x|] = 0 \text{ when } x \to 0.$$

13. (a, b)

$$\lim_{x\to 0^+} x\left(\frac{e^{|x|+[x]}-2}{|x|+[x]}\right)$$

$$= \lim_{x\to 0^+} x\left(\frac{e^{x+0}-2}{x+0}\right)$$

$$= \lim_{x\to 0^+} (e^x - 2)$$

$$= 1 - 2 = -1$$

$$\lim_{x\to 0^-} x\left(\frac{e^{|x|+[x]}-2}{|x|+[x]}\right)$$

$$= \lim_{x\to 0^-} x\left(\frac{e^{-x-1}-2}{-x-1}\right) = 0$$

14. (a, d)

$$\frac{\theta^2+\theta-2}{\theta+3} \le \frac{f(\theta)}{\theta^2} \le \frac{\theta^2+2\theta-1}{\theta+3}$$

Putting $\theta = -1$, we get

$$\frac{1-1-2}{2} \le f(-1) \le \frac{1-2-1}{2}$$

$\Rightarrow -1 \le f(-1) \le -1 \quad \Rightarrow \quad f(-1) = -1$

Also, $\lim_{\theta\to -1} \frac{\theta^2+\theta-2}{\theta+3} = -1 = \lim_{\theta\to -1} \frac{\theta^2+2\theta-1}{\theta+3}$

Using sandwich theorem

$$\lim_{\theta\to -1} \frac{f(\theta)}{\theta^2} = -1; \lim_{\theta\to -1} f(\theta) = -1$$

15. (a, c, d)

$\forall\ 1 < x < \frac{\pi}{2}$, $\tan^{-1}\tan x = x$

and $0 < \log_e x < \log_e \frac{\pi}{2} < 1$

$\Rightarrow \quad f(x) = x$

$\forall\ \frac{\pi}{2} < x < e$, $\tan^{-1}\tan x = x - \pi$

and $0 < \log_e x < 1$

$\therefore \quad (\log_e x)'' = 0$

$\Rightarrow \quad f(x) = x - \pi$

and for $x > e$, $\log_e x > 1$, $\therefore (\log_e x)^n \to \infty$

$\Rightarrow \quad f(x) = 0$

DPP 2.2

Single Correct Answer Type

1. (c) $\lim_{x\to 0} \frac{\sqrt{1-\cos x^2}}{1-\cos x}$

$$= \lim_{x\to 0} \frac{\sqrt{2\sin^2(x^2/2)}}{2\sin^2(x/2)}$$

$$= \frac{1}{\sqrt{2}} \lim_{x\to 0} \left(\frac{\frac{\sin(x^2/2)}{x^2/2}}{\left(\frac{\sin(x/2)}{x/2}\right)^2}\right) \times \frac{x^2/2}{x^2/4} = \sqrt{2}$$

2. (c) $\lim_{x\to \pi/2} (1-\sin x)\tan x$

$$= \lim_{x\to \pi/2} \sin x \frac{1-\sin x}{\cos x}$$

$$= \lim_{x\to \pi/2} \frac{(1-\sin^2 x)}{\cos x(1+\sin x)}$$

$$= \lim_{x\to \pi/2} \frac{\cos^2 x}{\cos x(1+\sin x)}$$

$$= \lim_{x\to \pi/2} \frac{\cos x}{(1+\sin x)} = 0$$

3. (c) $\lim_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right)$

$$= \lim_{x\to\infty} x^2 \times 2\sin^2\left(\frac{1}{2x}\right)$$

$$= \lim_{x\to\infty} \frac{2x^2 \times \sin^2\left(\frac{1}{2x}\right)}{\frac{1}{4x^2} \times 4x^2}$$

$$= \frac{1}{2}$$

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4. (d) $\lim_{x\to\infty}\sqrt[3]{x}\left(\sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}\right)$

$$= \lim_{x\to\infty} x^{1/3}\{(x+1)^{1/3}+(x-1)^{1/3}\}\{(x+1)^{1/3}-(x-1)^{1/3}\}$$

$$= \lim_{x\to\infty}\frac{x^{1/3}\{(x+1)^{1/3}+(x-1)^{1/3}\}2}{\{(x+1)^{2/3}+(x^2-1)^{1/3}+(x-1)^{2/3}\}}$$

$$= \lim_{x\to\infty}\frac{2x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{1/3}\right\}}{x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{2/3}+\left(1-\frac{1}{x^2}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{2/3}\right\}}=\frac{4}{3}$$

5. (d) $\lim_{n\to\infty}\frac{3\cdot 2^{n+1}-4\cdot 5^{n+1}}{5\cdot 2^n+7\cdot 5^n}$

$$= \lim_{n\to\infty}\frac{6\cdot 2^n-20\cdot 5^n}{5\cdot 2^n+7\cdot 5^n}$$

$$= \lim_{n\to\infty}\frac{6\cdot(2/5)^n-20}{5\cdot(2/5)^n+7}$$

$$=\frac{0-20}{0+7}=\frac{-20}{7}$$

6. (c) $\lim_{x\to 2^+}\{x\}\frac{\sin(x-2)}{(x-2)^2}$

$$= \lim_{x\to 2^+}\frac{\{x\}}{(x-2)}\frac{\sin(x-2)}{(x-2)}$$

$$= \lim_{x\to 2^+}\frac{x-[x]}{x-2}= \lim_{x\to 2^+}\frac{x-2}{x-2}=1$$

7. (a) $\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)=\lim_{x\to\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}}=0$

$\Rightarrow \quad \cot^{-1}(0)=\pi/2$

Also, $\lim_{x\to\infty}\left(\frac{2x+1}{x-1}\right)^x=\lim_{x\to\infty}\left(\frac{2+\frac{1}{x}}{1-\frac{1}{x}}\right)^x=\infty$

$\Rightarrow \quad \sec^{-1}(\infty)=\pi/2$

Thus, required limit is 1.

8. (d) $\lim_{x\to 0}\frac{3\tan 3x-4\tan 2x-\tan x}{4x^2\tan x}$

$$= \lim_{x\to 0}\frac{3[\tan 3x-\tan 2x]-(\tan 2x+\tan x)}{4x^2\tan x}$$

$$= \lim_{x\to 0}\frac{\frac{3\sin x}{\cos 2x\cos 3x}-\frac{\sin 3x}{\cos x\cos 2x}}{4x^2\tan x}$$

$$= \lim_{x\to 0}\frac{\frac{3}{\cos 2x\cos 3x}-\frac{3-4\sin^2 x}{\cos x\cos 2x}}{\frac{4x^2}{\cos x}}$$

$$= \lim_{x\to 0}\frac{1}{\cos 2x\cos 3x}\frac{3\cos x-(3-4\sin^2 x)\cos 3x}{4x^2}$$

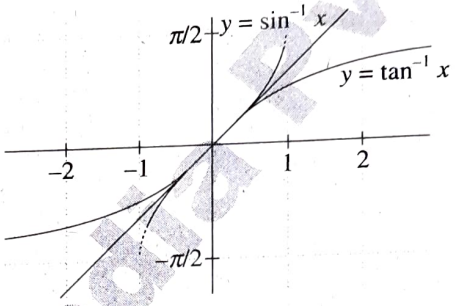
$$= \lim_{x\to 0}\frac{3(\cos x-\cos 3x)+4\sin^2 x}{4x^2}$$

$$= \lim_{x\to 0}\frac{6\sin x\sin 2x+4\sin^2 x}{4x^2}$$

$$= \lim_{x\to 0}\left(\frac{3}{2}\frac{\sin x}{x}\frac{2\sin 2x}{2x}+\frac{\sin^2 x}{x^2}\right)$$

$= 3+1=4$

9. (b)



$$L=\lim_{x\to 0}\left[\frac{\sin^{-1}x}{\tan^{-1}x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{\sin^{-1}x}{x}}{\frac{\tan^{-1}x}{x}}\right]$$

$$=\left[\frac{1\,(\text{but}>1)}{1\,(\text{but}<1)}\right]$$

$$=1$$

10. (d) $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi-4x}\times\frac{\sqrt{1+\sqrt{\sin 2x}}}{\sqrt{1+\sqrt{\sin 2x}}}$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{1-\sin 2x}}{(\pi-4x)}\times\lim_{x\to\frac{\pi}{4}}\frac{1}{\sqrt{1+\sqrt{\sin 2x}}}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{\sin^2\left(\frac{\pi}{4}-x\right)}}{\pi-4x}\cdot 1$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\left|\sin\left(\frac{\pi}{4}-x\right)\right|}{4\left(\frac{\pi}{4}-x\right)}$$

which gives RHL $=-\frac{1}{4}$ at $x=\frac{\pi}{4}$ and LHL $=\frac{1}{4}$ at $x=\frac{\pi}{4}$

So, limit does not exist.

11. (d) $L=\lim_{x\to\infty}\left(e^{\sqrt{x^4+1}}-e^{(x^2+1)}\right)$

$$= \lim_{x\to\infty}e^{x^2+1}\left[e^{\sqrt{x^4+1}-(x^2+1)}-1\right]$$

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Now, $\lim_{x\to\infty}\left[\sqrt{x^4+1}-(x^2+1)\right]$

$$=\lim_{x\to\infty}\frac{x^4+1-(x^4+1+2x^2)}{\sqrt{x^4+1}+(x^2+1)}$$

$$=\lim_{x\to\infty}\frac{-2x^2}{x^2\left(\sqrt{1+\left(1/x^4\right)}+1+\left(1/x^2\right)\right)}$$

$$=-1$$

$\therefore \quad L=\infty\times\left(\frac{1}{e}-1\right)$

$$=-\infty$$

12. (c) We have, $\lim_{x\to\pi/4}\frac{\tan^3 x-\tan x}{\cos(x+\pi/4)}$

$$=\lim_{x\to\pi/4}\frac{\tan x(\tan x-1)(\tan x+1)}{\cos(x+\pi/4)}$$

$$=\lim_{x\to\pi/4}\frac{\tan x(\sin x-\cos x)(\tan x+1)}{\cos x\cos(x+\pi/4)}$$

$$=-\lim_{x\to\pi/4}\frac{\tan x(\cos x-\sin x)(\tan x+1)}{\cos x\cos(x+\pi/4)}$$

$$=-\sqrt{2}\lim_{x\to\pi/2}\frac{\tan x\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)(\tan x+1)}{\cos x\cos(x+\pi/4)}$$

$$=-\sqrt{2}\lim_{x\to\pi/4}\frac{\tan x(\tan x+1)}{\cos x}$$

$$=-\sqrt{2}\times 2\times\sqrt{2}=-8$$

13. (d) $\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)(8x^3-\pi^3)\cos x}{(\pi-2x)^4}$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\left(1-\cos\left(\frac{\pi}{2}-x\right)\right)(2x-\pi)(4x^2+\pi^2+2\pi x)\sin\left(\frac{\pi}{2}-x\right)}{16\left(\frac{\pi}{2}-x\right)^4}$$

$$=\lim_{h\to 0}\frac{(1-\cos h)(-2h)\sin h}{16h^4}\lim_{x\to\frac{\pi}{2}}(4x^2+\pi^2+2\pi x)$$

$$=\lim_{h\to 0}\frac{(1-\cos h)(-2h)\sin h}{16h^4}\lim_{x\to\frac{\pi}{2}}(4x^2+\pi^2+2\pi x)$$

$$=-\lim_{h\to 0}\frac{(1-\cos h)}{8h^2}\left(4\left(\frac{\pi}{2}\right)^2+\pi^2+2\pi\left(\frac{\pi}{2}\right)\right)$$

$$=-(3\pi^2)\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{32\frac{h^2}{4}}$$

$$=-\frac{3\pi^2}{16}$$

14. (a) $\lim_{x\to\infty}\frac{\sum_{r=1}^{10}(x+r)^{2010}}{(x^{1006}+1)(2x^{1004}+1)}$

$$=\lim_{x\to\infty}\frac{(x+1)^{2010}+(x+2)^{2010}+\ldots+(x+10)^{2010}}{\left(x^{1006}+1\right)\left(2x^{1004}+1\right)}$$

$$=\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^{2010}+\left(1+\frac{2}{x}\right)^{2010}+\ldots+\left(1+\frac{10}{x}\right)^{2010}}{\left(1+\frac{1}{x^{1006}}\right)\left(2+\frac{1}{x^{1004}}\right)}$$

$$=\frac{10}{2}=5$$

15. (c) $\lim_{x\to 0}\frac{f(1-\cos x)}{g(x)\sin^2 x}$

$$=\lim_{x\to 0}\frac{f\left(2\sin^2\frac{x}{2}\right)}{g(x)\left(2\sin^2\frac{x}{2}\right)^2}\times\frac{\left(2\sin^2\frac{x}{2}\right)^2}{4\left(\sin^2\frac{x}{2}\right)\left(\cos^2\frac{x}{2}\right)}$$

$$=\lim_{x\to 0}\frac{a}{g(x)}\times\tan^2\frac{x}{2}$$

$$=a\lim_{x\to 0}\frac{\left(\frac{x}{2}\right)^2}{g(x)}\times\frac{\tan^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}$$

$$=a\lim_{x\to 0}\frac{x^2}{4g(x)}$$

$\therefore \quad \lim_{x\to 0}\frac{x^2}{g(x)}=\frac{4b}{a}$

Now, $\lim_{x\to 0}\frac{g(1-\cos 2x)}{x^4}$

$$=\lim_{x\to 0}\frac{g(2\sin^2 x)}{x^4}$$

$$=\lim_{x\to 0}\frac{g(2\sin^2 x)}{(2\sin^2 x)^2}\times\frac{(2\sin^2 x)^2}{x^4}$$

$$=\frac{a}{4b}\times 4=\frac{a}{b}$$

16. (b) $\lim_{x\to 0+} g(f(x))=g(f(0^+))=g(1^+)$ $\left(\because \lim_{x\to 0}\frac{x}{\sin x}=1^+\right)$

$$=1-2(1)-2=-3$$

$\lim_{x\to 0-} g(f(x))=g(f(0^-))=g(2^+)$ $\left(\because \lim_{x\to 0^-}(2-x)=2^+\right)$

$$=2-5=-3$$

$\Rightarrow \quad \lim_{x\to 0} g(f(x))=-3$

17. (b) $\lim_{n\to\infty}\left(\cos\frac{k\pi}{4}\right)^{2n}-\left(\cos\frac{k\pi}{6}\right)^{2n}=0$

holds good if

Case I: $\cos\frac{k\pi}{4}=\cos\frac{k\pi}{6}=1$

i.e., $\frac{k\pi}{4}=2m\pi$ and $\frac{k\pi}{6}=2p\pi$, $m, p\in Z$

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i.e., $k = 8m$ and $k = 12p$

i.e., k is divisible by both 8 and 12 i.e., k is divisible by 24.

Case II: $-1 < \cos\frac{k\pi}{4}, \cos\frac{k\pi}{6} < 1$

i.e., k is not divisible by 4 and k is not divisible by 6.

Case III: $\cos\frac{k\pi}{4} = -1 = \cos\frac{k\pi}{6}$

$\frac{k\pi}{4} = (2m+1)\pi$ and $\frac{k\pi}{6} = (2p+1)\pi$

$k = 4(2m+1)$ and $k = 6(2p+1)$

This is not possible.

18.(a) We have $a_n + \sqrt{2}b_n = (2+\sqrt{2})^n$

$\Rightarrow \quad a_n - \sqrt{2}b_n = (2-\sqrt{2})^n$

Solving we get $a_n = \frac{1}{2}[(2+\sqrt{2})^n + (2-\sqrt{2})^n]$

and $b_n = \frac{[(2+\sqrt{2})^n - (2-\sqrt{2})^n]}{2\sqrt{2}}$

$$\therefore \quad \frac{a_n}{b_n} = \sqrt{2}\frac{[(2+\sqrt{2})^n + (2-\sqrt{2})^n]}{[(2+\sqrt{2})^n - (2-\sqrt{2})^n]} = \sqrt{2}\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence, $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \sqrt{2}\left(\frac{1+0}{1-0}\right) = \sqrt{2} \qquad \left(\because \frac{2-\sqrt{2}}{2+\sqrt{2}} < 1\right)$

DPP 2.3

Single Correct Answer Type

1. (b) $$L = \lim_{x\to 0^+}\frac{\tan x^{\frac{1}{5}}}{(\tan^{-1}\sqrt{x})^2}\frac{\log(1+5x)}{(e^{\sqrt[5]{x}}-1)}$$

$$= \lim_{x\to 0^+}\frac{\tan\left(x^{\frac{1}{5}}\right)}{x^{\frac{1}{5}}}\cdot x^{\frac{1}{5}}\cdot\left(\frac{\sqrt{x}}{\tan^{-1}\sqrt{x}}\right)^2\cdot\frac{1}{x}\cdot 5x\cdot\frac{\frac{\log(1+5x)}{5x}}{\left(\frac{e^{3\cdot x^{\frac{1}{5}}}-1}{3x^{\frac{1}{5}}}\right)\cdot 3x^{\frac{1}{5}}}$$

$$= 1\cdot 1\cdot\frac{\log e}{1}\cdot\frac{5}{3} = \frac{5}{3}$$

2. (a) $$\lim_{x\to 3}\frac{(x^3+27)\log_e(x-2)}{(x^2-9)}$$

$$= \lim_{x\to 3}\frac{(x+3)(x^2+9-3x)\log_e(1+(x-3))}{(x+3)(x-3)}$$

$$= \lim_{x\to 3}(x^2+9-6x)$$

$$= 18-9 = 9$$

3. (d) $$\lim_{x\to 0^+}\left(\frac{1-\cos(\sin^2 x)}{x^2}\right)^{\frac{\log_e(1-2x^2)}{\sin^2 x}}.$$

As $\lim_{x\to 0^+}\left(\frac{1-\cos(\sin^2 x)}{\sin^4 x}\right)\times\frac{\sin^4 x}{x^2} = \frac{1}{2}\times 0 = 0$

and $\lim_{x\to 0^+}\frac{\left(\frac{\ln(1-2x^2)}{-2x^2}\right)\cdot(-2x^2)}{\left(\frac{\sin^2 x}{x^2}\right)\cdot x^2} = -2$

So, $\lim_{x\to 0^+}\left(\frac{1-\cos(\sin^2 x)}{x^2}\right)^{\frac{\ln(1-2x^2)}{\sin^2 x}} = \infty$

4. (b) $$\lim_{x\to 0}\frac{1}{x^2}\begin{vmatrix} 1-\cos 3x & \log_e(1+4x) \\ \sin^{-1}(e^x-1) & \tan^{-1}(2x)\end{vmatrix}$$

$$= \lim_{x\to 0}\begin{vmatrix} \frac{1-\cos 3x}{x} & \frac{\log_e(1+4x)}{x} \\ \frac{\sin^{-1}(e^x-1)}{x} & \frac{\tan^{-1}(2x)}{x}\end{vmatrix}$$

$$= \begin{vmatrix} \lim_{x\to 0}\frac{1-\cos 3x}{x} & \lim_{x\to 0}\frac{\log_e(1+4x)}{x} \\ \lim_{x\to 0}\frac{\sin^{-1}(e^x-1)}{x} & \lim_{x\to 0}\frac{\tan^{-1}(2x)}{x}\end{vmatrix}$$

$$= \begin{vmatrix} 0 & 4 \\ 1 & 2\end{vmatrix} = -4$$

5. (c) Since $y = f(x)$ is passing through the point $(3, 1)$, $f(3) = 1$.
Also, $y = f(x)$ is continuous.

$$\therefore \quad \lim_{x\to 3} f(x) = 1$$

$$\therefore \quad \lim_{x\to 3}\frac{\log_e(3f(x)-2)}{2(1-f(x))}$$

$$= \lim_{x\to 3}\frac{\log_e(1+3(f(x)-1))}{-2(f(x)-1)}$$

$$= -\frac{3}{2}$$

6. (a) $$\lim_{x\to 0}\left[f(x)+\log\left(1-\frac{1}{e^{f(x)}}\right)-\log(f(x))\right] = 0$$

$$\Rightarrow \quad \lim_{x\to 0}\left[f(x)+\log\left(\frac{e^{f(x)}-1}{e^{f(x)}}\right)-\log(f(x))\right] = 0$$

$$\Rightarrow \quad \lim_{x\to 0}\left[f(x)+\log\left(\frac{e^{f(x)}-1}{f(x)}\right)-f(x)\right] = 0$$

$$\Rightarrow \quad \log\left(\lim_{x\to 0}\left(\frac{e^{f(x)}-1}{f(x)}\right)\right) = 0$$

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$\Rightarrow \lim_{x\to 0}\left(\frac{e^{f(x)}-1}{f(x)}\right)=1$

$\Rightarrow \lim_{x\to 0} f(x)=0$

$\Rightarrow f(0)=0$

7. (c) $\lim_{x\to\infty} x^2 \sin\left(\log_e\sqrt{\cos\frac{\pi}{x}}\right)$ (Form $0\times\infty$)

$$= \lim_{x\to\infty} \frac{x^2 \sin\left(\log_e\sqrt{\cos\frac{\pi}{x}}\right)}{\left(\log_e\sqrt{\cos\frac{\pi}{x}}\right)}\left(\log_e\sqrt{\cos\frac{\pi}{x}}\right)$$

$$= \lim_{x\to\infty} x^2.\log_e\sqrt{\cos\frac{\pi}{x}}$$

$$= \lim_{x\to\infty} \frac{x^2}{2}\log_e\left(\cos\frac{\pi}{x}\right)$$

$$= \lim_{x\to\infty} \frac{x^2}{2}\frac{\log_e\left[1+\left(\cos\frac{\pi}{x}-1\right)\right]}{\left(\cos\frac{\pi}{x}-1\right)}\left(-2\sin^2\frac{\pi}{2x}\right)$$

$$= \lim_{x\to\infty} \frac{x^2}{2}\frac{\left(-2\sin^2\frac{\pi}{2x}\right)}{\left(\frac{\pi}{2x}\right)^2}.\left(\frac{\pi}{2x}\right)^2$$

$$= -\frac{\pi^2}{4}$$

8. (d) $\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x = 4$

$\Rightarrow e^{\lim_{x\to\infty}\left[\left(\frac{x+c}{x-c}-1\right)\cdot x\right]}=4$

$\Rightarrow e^{\lim_{x\to\infty}\left[\left(\frac{x+c-x+c}{x-c}\right)\cdot x\right]}=4$

$\Rightarrow e^{2c}=4$

$\Rightarrow 2c=\log 4 \Rightarrow c=\log 2$

$\Rightarrow e^c=2$

9. (b) $\lim_{x\to 0}\left[1+x+\frac{f(x)}{x}\right]^{1/x}=e^3$

$\Rightarrow \lim_{x\to 0} e^{\lim_{x\to 0}\left[1+x+\frac{f(x)}{x}-1\right]\frac{1}{x}}=e^3$

$\Rightarrow \lim_{x\to 0} e^{\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]}=e^3$

$\Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2}=2$

Now, $\lim_{x\to 0}\left[1+\frac{f(x)}{x}\right]^{1/x}=e^{\lim_{x\to 0}\left[1+\frac{f(x)}{x}-1\right]\frac{1}{x}}=e^{\lim_{x\to 0}\frac{f(x)}{x^2}}=e^2$

10. (d) $L=\lim_{x\to\frac{\pi}{2}}|1+(\cos x)^{\cos x}|^2$

Let $y=\lim_{x\to\frac{\pi}{2}^-}(\cos x)^{\cos x}$

$\therefore \log(y)=\lim_{x\to\frac{\pi}{2}^-}(\cos x)\log\cos x$

$\therefore \log(y)=\lim_{x\to\frac{\pi}{2}^-}\frac{\log(\cos x)}{\sec(x)}$ (∞/∞)

Using L'Hospital's rule, we get

$$\log(y)=\lim_{x\to\frac{\pi}{2}^-}\left[-\left(\frac{1}{\cos x}\times\frac{\sin x}{\sec x\tan x}\right)\right]$$

$\Rightarrow \log(y)=\lim_{x\to\frac{\pi}{2}^-}(-\cos x)=0$

$\therefore y=e^0=1$

$\therefore L=(1+1)^2=2^2=4$

11. (a) $L=\lim_{n\to\infty}\left(\frac{a-1+b^{\frac{1}{n}}}{a}\right)^n$

Let $\frac{1}{n}=x,$

$\therefore L=\lim_{x\to 0}\left(\frac{a-1+b^x}{a}\right)^{\frac{1}{x}}$

$$= e^{\lim_{x\to 0}\left(\frac{a-1+b^x}{a}-1\right)\frac{1}{x}}$$

$$= e^{\lim_{x\to 0}\left(\frac{b^x-1}{ax}\right)}$$

$$= e^{\frac{1}{a}\log_e b} = b^{\frac{1}{a}}$$

12. (a) $f(x)=\lim_{n\to\infty}\left(\cos\frac{x}{\sqrt{n}}\right)^n$

$$= e^{\lim_{n\to\infty}\left(\cos\frac{x}{\sqrt{n}}-1\right)\cdot n}$$

$$= e^{-\left(2\lim_{n\to\infty}\frac{\frac{x^2}{4}\sin^2\frac{x}{2\sqrt{n}}}{\frac{x^2}{4\sqrt{n}}}\right)}$$

$$= e^{-\frac{1}{2}x^2}$$

$\Rightarrow \lim_{x\to 0}\frac{f(x)-1}{x}=\lim_{x\to 0}\frac{e^{-\frac{1}{2}x^2}-1}{x}=0$

13. (d) $\lim_{x\to 0}\frac{\log(e^{x^2}+2\sqrt{x})}{\tan\sqrt{x}}$

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$$= \lim_{x\to 0} \frac{\log\left(e^{x^2} + 2\sqrt{x}\right)}{\frac{\tan\sqrt{x}}{\sqrt{x}}\sqrt{x}}$$

$$= \lim_{x\to 0} \frac{\log\left(e^{x^2} + 2\sqrt{x}\right)}{\sqrt{x}}$$

$$= \lim_{x\to 0} \log\left(e^{x^2} + 2\sqrt{x}\right)^{\frac{1}{\sqrt{x}}}$$

$$= \log e^{\lim_{x\to 0}\left(e^{x^2}+2\sqrt{x}-1\right)\frac{1}{\sqrt{x}}}$$

$$= \log e^{\lim_{x\to 0}\left(\frac{e^{x^2}-1}{\sqrt{x}}+2\right)}$$

$$= 2$$

14. (b) $\lim_{x\to 0}\left(\frac{f^2(a+x)}{f(a)}\right)^{1/x}$

$$= e^{\lim_{x\to 0}\left(\frac{f^2(a+x)-f(a)}{xf(a)}\right)}$$

$$= e^{\lim_{x\to 0}\frac{f^2(a+x)-(f(a))^2}{x}}$$

$$= e^{\lim_{x\to 0}\frac{(f(a+x)-f(a))(f(a+x)+f(a))}{x}}$$

$$= e^{\lim_{x\to 0} 2f(a)\frac{(f(a+x)-f(a))}{x}}$$

$$= e^{\lim_{x\to 0} 2f(a)f'(a)}$$

$$= e^4$$

15. (b) $L = \lim_{n\to\infty}\left(\frac{\sqrt{n^2+n}-1}{n}\right)^{2\sqrt{n^2+n}-1}$

$$= e^{\lim_{n\to\infty}\left[\frac{\sqrt{n^2+n}-1-n}{n}\right](2\sqrt{n^2+n}-1)}$$

$$= e^{\lim_{n\to\infty}\frac{[(n^2+n)-(1+n)^2](2\sqrt{n^2+n}-1)}{n\left\{\sqrt{n^2+n}+(1+n)\right\}}}$$

$$= e^{\lim_{n\to\infty}\frac{(-n-1)(2\sqrt{n^2+n}-1)}{n\left\{\sqrt{n^2+n}+(1+n)\right\}}}$$

$$= e^{\lim_{n\to\infty}\frac{\left(-1-\frac{1}{n}\right)\left(2\sqrt{1+\frac{1}{n}}-\frac{1}{n}\right)n^2}{n^2\left\{\sqrt{1+\frac{1}{n}}+\frac{1}{n}+1\right\}}} = e^{-1}$$

16. (b) $f(n) = \lim_{x\to 0} e^{\frac{1}{x}\left(\left(1+\sin\frac{x}{2}\right)\left(1+\sin\frac{x}{2^2}\right)\cdots\left(1+\sin\frac{x}{2^n}\right)-1\right)}$

$$= \lim_{x\to 0} e^{\frac{\left(1+\left(\sin\frac{x}{2}+\sin\frac{x}{2^2}+\ldots+\sin\frac{x}{2^n}\right)+\left(\sin\frac{x}{2}\sin\frac{x}{2^2}+\ldots\right)+\ldots-1\right)}{x}}$$

$$= \lim_{x\to 0} e^{\left(\frac{\sin\frac{x}{2}}{x}+\frac{\sin\left(\frac{x}{2^2}\right)}{x}+\ldots+\sin\frac{\left(\frac{x}{2^n}\right)}{x}\right)}$$

$$= e^{\left(\frac{1}{2}+\frac{1}{2^2}+\ldots\frac{1}{2^n}\right)}$$

$$\therefore \lim_{n\to\infty} f(n) = e^{1-\frac{1}{2}^{1/2}} = e$$

17. (a) $\lim_{n\to\infty}(1-x+x\cdot\sqrt[n]{e})^n$ (1^∞ form)

$$= e^{\lim_{n\to\infty} n\cdot(1-x+xe^{1/n}-1)}$$

$$= e^x e^{\lim_{n\to\infty}\frac{(e^{1/n}-1)}{1/n}}$$

$$= e^x$$

DPP 2.4

Single Correct Answer Type

1. (b) $\lim_{x\to 1}\frac{x^{\frac{1}{13}}-x^{\frac{1}{7}}}{x^{\frac{1}{5}}-x^{\frac{1}{3}}}$ $\left(\frac{0}{0}\text{ form}\right)$

Apply L'Hospital's rule

$$= \lim_{x\to 1}\frac{\frac{1}{13}x^{-1+\frac{1}{13}}-\frac{1}{7}x^{-1+\frac{1}{7}}}{\frac{1}{5}x^{\frac{1}{5}-1}-\frac{1}{3}x^{\frac{1}{3}-1}}$$

$$= \frac{\frac{1}{13}-\frac{1}{7}}{\frac{1}{5}-\frac{1}{3}} = \frac{45}{91}$$

2. (d) Applying L'Hospital's rule,

$$\lim_{x\to 1}\frac{1+\log x-x}{1-2x+x^2} = \lim_{x\to 1}\frac{\frac{1}{x}-1}{-2+2x} = \lim_{x\to 1}\frac{1-x}{2x(x-1)}$$

$$= \lim_{x\to 1}\frac{-1}{2x} = -\frac{1}{2}$$

3. (c) Let $L = \lim_{x\to 0}\frac{e^{x^2}-e^x+x}{1-\cos 2x}$

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$$= \lim_{x\to 0} \frac{e^{x^2} - 1 + x - e^x + 1}{2\frac{\sin^2 x}{x^2}\cdot x^2}$$

$$= \frac{1}{2}\left[\lim_{x\to 0}\frac{e^{x^2}-1}{x^2} + \lim_{x\to 0}\frac{x - e^x + 1}{x^2}\right]$$

$$= \frac{1}{2}\left[1 - \lim_{x\to 0}\frac{e^x - 1}{2x}\right] \text{(using L' Hospital Rule)}$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}\right]$$

$$= \frac{1}{4}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{1-\cos 2x}{e^{x^2} - e^x + x} = 4$$

4. (c) $\lim_{h\to 0}\dfrac{f(a+2h^2) - f(a-2h^2)}{f(a+h^3-h^2) - f(a-h^3+h^2)}$

$$= \lim_{h\to 0}\frac{f'(a+2h^2)\,4h - f'(a-2h^2)(-4h)}{f'(a+h^3-h^2)(3h^2-2h) - f'(a-h^3+h^2)(-3h^2+2h)}$$

$$= \lim_{h\to 0}\frac{4h}{3h^2-2h}\left[\frac{f'(a+2h^2)+f'(a-2h^2)}{f'(a+h^3-h^2)+f'(a-h^3+h^2)}\right]$$

$$= \lim_{h\to 0}\frac{4}{3h-2} = -2$$

5. (d) $\lim_{x\to 0^+}\dfrac{1}{x\sqrt{x}}\left(a\tan^{-1}\dfrac{\sqrt{x}}{a} - b\tan^{-1}\dfrac{\sqrt{x}}{b}\right)$

$$= \lim_{x\to 0}\frac{\dfrac{a}{\left(1+\dfrac{x}{a^2}\right)}\cdot\dfrac{1}{a}\cdot\dfrac{1}{2\sqrt{x}} - \dfrac{b}{\left(1+\dfrac{x}{b^2}\right)}\cdot\dfrac{1}{b}\cdot\dfrac{1}{2\sqrt{x}}}{\dfrac{3}{2}\cdot\sqrt{x}}$$

$$= \lim_{x\to 0}\left[\frac{a^2}{(a^2+x)x} - \frac{b^2}{(b^2+x)x}\right]\cdot\frac{1}{3}$$

$$= \frac{a^2-b^2}{3a^2b^2}$$

6. (a) $\lim_{x\to 0} e^{\frac{1}{x^2}(\log(1+2x) - \log(1+3x) + \frac{1}{x})}$

$$= e^{\lim_{x\to 0}\frac{(\log(1+2x)-\log(1+3x)+x}{x^2}}$$

$$= e^{\lim_{x\to 0}\frac{2x - \frac{(2x)^2}{2} - 3x + \frac{(3x)^2}{2} + x}{x^2}} \text{(using expansion)}$$

$$= e^{\lim_{x\to 0}\frac{\frac{5}{2}x^2}{x^2}}$$

$$= e^{\frac{5}{2}}$$

7. (b) $L = \lim_{x\to 0}\sum_{n=1}^{\infty}\dfrac{(-1)^n f\left(\frac{x}{n}\right)}{x}$

$$= \lim_{x\to 0}\sum_{n=1}^{\infty}\frac{(-1)^n\frac{1}{n}f'\left(\frac{x}{n}\right)}{1} \quad \text{(Applying L'Hospital Rule)}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n\frac{1}{n}f'(0)}{1}$$

$$= -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

$$= -\log 2$$

8. (d) $L = \lim_{x\to 3\pi/4}\dfrac{1+(\tan x)^{\frac{1}{3}}}{1-2\cos^2 x}$

$$= \lim_{x\to 3\pi/4}\frac{1+(\tan x)^{\frac{1}{3}}}{-\cos 2x} \quad \left(\frac{0}{0}\text{form}\right)$$

Applying L'Hospital's rule

$$\therefore \quad L = \lim_{x\to\frac{3\pi}{4}}\frac{\frac{1}{3}(\tan x)^{\frac{-2}{3}}\times\sec^2 x}{2\sin 2x}$$

$$= \frac{-1}{6}\left(-\sec\frac{\pi}{4}\right)^2$$

$$= \frac{-1}{6}\times 2 = -\frac{1}{3}$$

9. (c) We have $\lim_{x\to 1^+}\dfrac{(x-1)^n}{\log\cos^m(x-1)} = -1$

$$\Rightarrow \quad \lim_{h\to 0}\frac{h^n}{m(\log\cos h)} = -1$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{nh^{n-2}\cos h}{m\left(\frac{\sin h}{h}\right)} = 1 \quad \text{(Applying L' Hospital Rule)}$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{nh^{n-2}}{m} = 1$$

$$\Rightarrow \quad n = 2 \text{ and } m = 2$$

Comprehension Type

10.(c), 11.(c)

Since $\lim_{x\to 1}\dfrac{f(x)}{(x-1)^2} = 1,\ f(1) = 0$

$$\therefore \quad \lim_{x\to 1}\frac{f(x)}{(x-1)^2} = \lim_{x\to 1}\frac{f'(x)}{2(x-1)} = 1$$

$$\Rightarrow \quad f'(1) = 0$$

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$\therefore \quad \lim_{x\to 1}\frac{f''(x)}{2} = 1 \Rightarrow f''(1) = 2$

Since $x = 1$ is root of $f(x) = 0$ and $f'(x) = 0$,

$f(x) = (x-1)^2(ax^2+bx+2) \qquad (\because f(0) = 2)$

$\Rightarrow \quad f'(x) = 2(x-1)(ax^2+bx+2) + (2ax+b)(x-1)^2$

$\because \quad f'(0) = -6 \Rightarrow b = -2$

Using $f''(1) = 2$, we get $a + b = -1 \Rightarrow a = 1$

$\Rightarrow \quad f(x) = (x-1)^2(x^2-2x+2)$

$\Rightarrow \quad f(x) = (x-1)^4 + (x-1)^2$

$\Rightarrow \quad f(2) = 1 + 1 = 2$

Also, $f'(x) = 4(x-1)^3 + 2(x-1)$

$\Rightarrow \quad f'(2) = 4 + 2 = 6$

DPP 2.5

Single Correct Answer Type

1. (c) $L = \lim_{x\to 1}\sec^{-1}\left(\frac{\lambda^2}{\log_e x} - \frac{\lambda^2}{x-1}\right)$

let $x - 1 = t$

$\therefore \quad L = \lim_{t\to 0}\sec^{-1}\left(\frac{\lambda^2}{\ln(t+1)} - \frac{\lambda^2}{t}\right)$

$= \lim_{t\to 0}\sec^{-1}\left(\lambda^2\left(\frac{t-\ln(t+1)}{t\ln(t+1)}\right)\right)$

$= \lim_{t\to 0}\sec^{-1}\left(\lambda^2\left(\frac{t-\left(t-\frac{t^2}{2}+\ldots\right)}{t\ln(t+1)}\right)\right)$

$= \lim_{t\to 0}\sec^{-1}\left(\lambda^2\left(\frac{\frac{t^2}{2}-\frac{t^3}{3}\ldots}{t\ln(t+1)}\right)\right)$

$= \lim_{t\to 0}\sec^{-1}\left(\lambda^2\left(\frac{\frac{t}{2}-\frac{t^2}{3}\ldots}{\ln(t+1)}\right)\right)$

$= \lim_{t\to 0}\sec^{-1}\left(\frac{\lambda^2}{2}\right)$ exists

$\Rightarrow \frac{\lambda^2}{2} \geq 1 \Rightarrow |\lambda| \geq \sqrt{2}$

Thus, L does not exist for $\lambda = 0, \pm 1$.

2. (c) $L = \lim_{x\to 0}\frac{e^{ax} - e^x - x}{x^2}$

$= \lim_{x\to 0}\frac{ae^{ax} - e^x - 1}{2x}$ (Using L'Hopital's Rule)

Since denominator $\to 0$, we must have numerator $\to 0$ when $x \to 0$

$\therefore \quad ae^0 - e^0 - 1 = 0$ or $a = 2$

$\therefore \quad L = \lim_{x\to 0}\frac{2e^{2x} - e^x - 1}{2x}$

$= \lim_{x\to 0}\frac{4e^{2x} - 1}{2}$ (Using L'Hopital's Rule)

$= \frac{3}{2}$

3. (b) $\lim_{x\to 0}\frac{x^3}{\sqrt{a+x}\,(bx - \sin x)} = 1$

$\Rightarrow \quad \lim_{x\to 0}\frac{x^3}{\sqrt{a}\,(bx - \sin x)} = 1$

$\Rightarrow \quad \lim_{h\to 0}\frac{3x^2}{\sqrt{a}\,(b - \cos x)} = 1$

$\therefore \quad b = 1$

$\therefore \quad 1 = \lim_{x\to 0}\frac{3x^2}{\sqrt{a}\,(1-\cos x)}$

$= \lim_{x\to 0}\frac{3x^2}{\sqrt{a}\,2\sin^2\frac{x}{2}} = \frac{6}{\sqrt{a}}$

$\therefore \quad a = 36$

Thus $a + b = 37$

4. (d) Let, $L = \lim_{x\to\infty} x\cdot\log_e\left(\begin{vmatrix} \alpha/x & 1 & \gamma \\ 0 & 1/x & \beta \\ 1 & 0 & 1/x \end{vmatrix}\right)$

$= \lim_{x\to\infty} x\cdot\log_e\left(\frac{\alpha}{x^3} + \beta - \frac{\gamma}{x}\right)$

For limit to exist, $\beta = 1$

$\therefore \quad L = \lim_{x\to\infty} x.\log_e\left(1 + \frac{\alpha}{x^3} - \frac{\gamma}{x}\right)$

$= \lim_{x\to\infty} x\left(\frac{\alpha}{x^3} - \frac{\gamma}{x}\right)$

$= \lim_{x\to\infty}\left(\frac{\alpha}{x^2} - \gamma\right) = -\gamma = -5$ and $\alpha \in R$

Multiple Correct Answers Type

5. (a, b, c)

$\lim_{x\to 0}\frac{ae^x + b\cos x + ce^{-x}}{e^{2x} - 2e^x + 1} = 4$

$\Rightarrow \lim_{x\to 0}\frac{a\left(1+x+\frac{x^2}{2}+\ldots\right) + b\left(1-\frac{x^2}{2}\ldots\right) + c\left(1-x+\frac{x^2}{2}\right)}{x^2}\cdot\left(\frac{x}{e^x-1}\right)^2 = 4$

$\Rightarrow \quad a + b + c = 0,\ a - c = 0 \Rightarrow a = c$

$\Rightarrow \quad \frac{a - b + c}{2} = 4$

$\Rightarrow \quad a - b + c = 8 \Rightarrow 2b = -8 \Rightarrow b = -4$

6. **(a, b)**

Let $f(x) = [x^3] - [x]^3$

$f(a) = 0$

$f(a^+) = a^3 - a^3 = 0$

$f(a^-) = \lim_{h\to 0} ([(a-h)^3] - [a-h]^3) = a^3 - 1 - (a-1)^3 = 3a(a-1)$

Since $f(a^-) = 0 \Rightarrow 3a(a-1) = 0 \Rightarrow a = 0$ or $a = 1$

Comprehension Type

7. **(a), 8.** **(b)**

We have $\lim_{x\to 0} \frac{\sin(\sin x) - \sin x}{ax^5 + bx^3 + c} = -\frac{1}{12}$

In L.H.S. $N^r \to 0$ when $x \to 0$

$\therefore$ D^r must $\to 0$

$\Rightarrow c = 0$

Now, $\lim_{x\to 0} \frac{\sin(\sin x) - \sin x}{ax^5 + bx^3}$

$$= \lim_{x\to 0} \frac{2\sin\left(\frac{\sin x - x}{2}\right)\cos\left(\frac{\sin x + x}{2}\right)}{ax^5 + bx^3}$$

$$= \lim_{x\to 0} \frac{2\sin\left(\frac{\sin x - x}{2}\right)}{\left(\frac{\sin x - x}{2}\right)} \cdot \frac{\sin x - x}{2} \cdot \frac{\cos\left(\frac{\sin x + x}{2}\right)}{ax^5 + bx^3}$$

$$= \lim_{x\to 0} \left(\frac{\sin x - x}{ax^5 + bx^3}\right)$$

$$= \lim_{x\to 0} \frac{\cos x - 1}{5ax^4 + 3bx^2}$$

$$= \lim_{x\to 0} \frac{-\sin x}{20ax^3 + 6bx} \quad \text{(Using L'Hospital's Rule)}$$

$$= \lim_{x\to 0} \frac{-\sin x}{x} \cdot \frac{1}{20ax^2 + 6b}$$

$$= -\frac{1}{6b} \text{ for all } a \in R$$

$\therefore$ $-\frac{1}{6b} = -\frac{1}{12} \Rightarrow b = 2$

$\therefore$ $a \in R, b = 2, c = 0$

9. **(b), 10.** **(c)**

$$f(x) = \lim_{n\to\infty} \frac{(x^2 + ax + 1) + x^{2n}(2x^2 + x + b)}{1 + x^{2n}}$$

$$= \begin{cases} x^2 + ax + 1, & |x| < 1 \\ 2x^2 + x + b, & |x| > 1 \\ \frac{-a + b + 3}{2}, & x = -1 \\ \frac{a + b + 5}{2}, & x = 1 \end{cases}$$

$\lim_{x\to -1} f(x)$ exists if

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$$

$$\Rightarrow \lim_{x\to -1^-} (2x^2 + x + b) = \lim_{x\to -1^+} (x^2 + ax + 1)$$

$$\Rightarrow 2 - 1 + b = 1 - a + 1$$

$$\Rightarrow a + b = 1 \quad \text{(i)}$$

$\lim_{x\to 1} f(x)$ exists if

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$\Rightarrow \lim_{x\to 1^-} (x^2 + ax + 1) = \lim_{x\to 1^+} (2x^2 + x + b)$$

$$\Rightarrow 1 + a + 1 = 2 + 1 + b$$

$$\Rightarrow a - b = 1 \quad \text{(ii)}$$

Solving Eqs. (i) and (ii), we get $a = 1$ and $b = 0$.

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CHAPTER 3

DPP 3.1

Single Correct Answer Type

1. (b) $f(x)$ is continuous at $x = 0$

$$\therefore \quad \lim_{x\to 0}\left(\sin\frac{2x^2}{a}+\cos\frac{3x}{b}\right)^{ab/x^2} = f(0)$$

$$\Rightarrow \quad e^3 = e^{\lim_{x\to 0}\left(\sin\frac{2x^2}{a}+\cos\frac{3x}{b}-1\right)\frac{ab}{x^2}}$$

$$\Rightarrow \quad 3 = ab\lim_{x\to 0}\left(\frac{2}{a}\frac{\sin\frac{2x^2}{a}}{\frac{2x^2}{a}}-\frac{9}{4b^2}\frac{2\sin^2\frac{3x}{2b}}{\frac{9x^2}{4b^2}}\right)$$

$$\Rightarrow \quad 3 = ab\left(\frac{2}{a}-\frac{9}{2b^2}\right)$$

$$\Rightarrow \quad 3 = ab\left(\frac{4b^2-9a}{2ab^2}\right)$$

$$\Rightarrow \quad 3 = \frac{4b^2-9a}{2b}$$

$$\Rightarrow \quad 4b^2 - 6b - 9a = 0$$

Since b is real, we have

$$D \geq 0$$

$$\Rightarrow \quad 36 + 144a \geq 0$$

$$\Rightarrow \quad a \geq -1/4$$

2. (c) (a) Since $g(x) = |f(x)|$

$\therefore$ $g(x) \geq 0$

$\therefore$ Range of $g \neq R$. Hence, g is not onto.

(b) If we take $f(x) = x$, then f is one-one but $|f(x)| = |x|$ is not one-one.

(c) If $f(x)$ is continuous then $|f(x)|$ is also continuous.

3. (d) $f\left(\frac{1}{2}\right) = p$ (i)

$$\lim_{x\to\frac{1}{2}^-}\frac{1-\sin\pi x}{1+\cos(2\pi x)}$$

$$= \lim_{x\to\frac{1}{2}^-}\frac{-\pi\cos(\pi x)}{-2\pi\sin(2\pi x)} \quad \text{(Using L'Hospital's Rule)}$$

$$= \lim_{x\to\frac{1}{2}^-}\frac{-\pi\sin(\pi x)}{4\pi\cos(2\pi x)} \quad \text{(Using L'Hospital's Rule)}$$

$$= \frac{1}{4}$$

$$\lim_{x\to\left(\frac{1}{2}\right)^+}\left(\frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}\right)$$

$$= \lim_{x\to\frac{1}{2}^+}\left(\frac{\sqrt{2x-1}}{4+\sqrt{2x-1}-4}\right)\left(\sqrt{4+\sqrt{2x-1}}+2\right)$$

$$= 4$$

Since L.H.L. $\neq$ R.H.L., value of p does not exist.

4. (d)

5. (c) $h(x) = \frac{f(x)}{g(x)}$

$$\Rightarrow \quad h(x) = \frac{x^3-x^2-3x-1}{(x+1)a} \quad \text{(i)}$$

Now, $\lim_{x\to -1} h(x) = \frac{1}{2}$

$$\Rightarrow \quad \lim_{x\to -1}\frac{x^3-x^2-3x-1}{(x+1)a} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{2}{a} = \frac{1}{2} \Rightarrow a = 4$$

$$\therefore \quad h(x) = \frac{x^3-x^2-3x-1}{4(x+1)}$$

$$\therefore \quad h(1) = -1/2$$

6. (b) Since the function is continuous at $x = -2$, then

$$f(-2) = \lim_{x\to -2} f(x)$$

$$= \lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$$

this limit will exist if $15 - a = 0$

$$\Rightarrow \quad a = 15 \quad \text{(i)}$$

So, $$\lim_{x\to -2} f(x) = \lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2}$$

$$= \lim_{x\to -2}\frac{3(x+2)(x+3)}{(x+2)(x-1)}$$

$$= \lim_{x\to -2}\frac{3(x+3)}{(x-1)}$$

$$= \frac{3}{-3} = -1$$

Hence, $f(-2) = -1$.

7. (c) $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} 8^{-\frac{1}{h}} = 0$ (where $h > 0$)

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} a[x] = a(0) = 0$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

At positive integral points, $f(x)$ is not continuous.

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8. (b) $f(x) = \begin{cases} \dfrac{2\cos x - \sin 2x}{(\pi - 2x)^2}, & x \le \dfrac{\pi}{2} \\ \dfrac{e^{-\cos x} - 1}{8x - 4\pi}, & x > \dfrac{\pi}{2} \end{cases}$

L.H.L. at $x = \pi/2$

$$\lim_{h\to 0} \frac{2\sin h - \sin 2h}{4h^2} = \lim_{h\to 0} \frac{2\sin h(1-\cos h)}{4h^2} = 0$$

$$\text{R.H.L.} = \lim_{h\to 0} \frac{e^{\sin h} - 1}{8((\pi/2) + h) - 4\pi}$$

$$= \lim_{h\to 0} \frac{e^{\sin h} - 1}{8h} \cdot \frac{\sin h}{\sin h} = \frac{1}{8}$$

$\Rightarrow$ $h(x)$ has irremovable discontinuity at $x = \pi/2$.

9. (a) $f(5^-) = \lim_{x\to 5^-} \sin\left(\frac{\pi}{2}(x - [x])\right)$

$$= \lim_{x\to 5^-} \sin\left(\frac{\pi}{2}(x-4)\right) = 1$$

$$f(5^+) = \lim_{x\to 5^+} \frac{ab^2|x^2 - 11x + 24|}{x-3}$$

$$= \lim_{x\to 5^+} \frac{-ab^2(x-3)(x-8)}{x-3} = 3ab^2$$

$\Rightarrow$ $5(b-1) = 1 = 3ab^2$

$\Rightarrow$ $b = \frac{6}{5}, a = \frac{25}{108}$

10. (c) (i) $f(x) = \begin{cases} 1, & x \le 0 \\ -1, & x > 0 \end{cases}$

(ii) $f(x) = \begin{cases} 1, & x < 0 \\ -1, & x \ge 0 \end{cases}$

(iii) $f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases}$

(iv) $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \ge 0 \end{cases}$

(v) $f(x) = \begin{cases} 1, & x > 0 \\ 1, & x < 0 \\ -1 & x = 0 \end{cases}$

(vi) $f(x) = \begin{cases} -1, & x > 0 \\ -1, & x < 0 \\ 1 & x = 0 \end{cases}$

11. (b) $k = \lim_{x\to 2^+} (x^2 + e^{\frac{1}{2-x}})^{-1}$

$\therefore$ $k = \lim_{h\to 0} [(2+h)^2 + e^{\frac{-1}{h}}]^{-1}$

$= (4 + e^{-\infty})^{-1}$

$= 1/4$

12. (c) $f(x)$ is discontinuous at $x = 2$,

So, $f(f(x))$ may be discontinuous when $1 + x = 2$ (when $0 \le x \le 2$) or $3 - x = 2$ (when $2 < x \le 3$).

$\therefore$ $x = 1$

Hence $f(f(x))$ is discontinuous at $x = 1$ and $x = 2$.

13. (c) If f is continuous at $x = 0$, then

$$f(0) = \lim_{x\to 0} \frac{(128a + ax)^{1/8} - 2}{(32 + bx)^{1/5} - 2}$$

As $x \to 0$, the denominator $\to 0$.

Thus, for limit to exist the numerator must also $\to 0$.

Thus, we have $(128a)^{1/8} = 2$ or $a = 2$.

Now, we have

$$f(0) = \lim_{x\to 0} \frac{(256 + 2x)^{1/8} - 2}{(32 + bx)^{1/5} - 2} \qquad \left(\frac{0}{0}\right)$$

$$\Rightarrow \quad f(0) = \lim_{x\to 0} \frac{\frac{2}{8}(256 + 2x)^{-7/8}}{\frac{b}{5}(32 + bx)^{-4/5}} = \frac{5}{4b} \cdot \frac{2^{-7}}{2^{-4}} = \frac{5}{32b}$$

$\Rightarrow$ $b = \dfrac{5}{32 f(0)}$

Hence, we have $\frac{a}{b} = \frac{64}{5} f(0)$

14. (c) $f(x) = \begin{cases} \dfrac{1 - \cos\left(1 - \cos\frac{x}{2}\right)}{2^m x^n}, & x \ne 0 \text{ is continuous at } x = 0 \\ 1, & x = 0 \end{cases}$

Hence, $\lim_{x\to 0} \dfrac{\left[1 - \cos\left(1 - \cos\frac{x}{2}\right)\right]\left(1 - \cos\frac{x}{2}\right)^2}{\left(1 - \cos\frac{x}{2}\right)^2 \cdot 2^m \cdot x^n} = 1$

$\Rightarrow$ $\lim_{x\to 0} \dfrac{\left(1 - \cos\frac{x}{2}\right)^2}{2^m x^n} = 2 \quad \left[\text{Using } \lim_{\theta\to 0} \frac{1 - \cos\theta}{\theta^2} = \frac{1}{2}\right]$

$\Rightarrow$ $\lim_{x\to 0} \dfrac{4\sin^4\frac{x}{4}}{2^m x^n} = 2$

$\Rightarrow$ $\lim_{x\to 0} \dfrac{4x^4}{4^4 2^m x^n} = 2$

For limit to exist $n = 4$.

$\therefore$ $\dfrac{2}{2^{8+m}} = 1$

$\therefore$ $2^{8+m} = 2$

$\therefore$ $m = -7$

$\Rightarrow$ $m + n = -3$

Multiple Correct Answers Type

15. **(b, c, d)**

(a) Clearly, $f(x)$ is continuous at $x = 1$.

(b) $g(1^+) = 0$, $g(1) = 1 \Rightarrow g(x)$ is discontinuous at $x = 1$

(c) $h(1^+) = 1$ and $h(1^-) = 0 \Rightarrow h(x)$ is discontinuous at $x = 1$

(d) $\phi(1^+) = 1$ and $\phi(1^-) = -1 \Rightarrow \phi(x)$ is discontinuous at $x = 1$

16. **(c, d)**

Given f is continuous in $[a, b]$ (i)

g is continuous in $[b, c]$ (ii)

$f(b) = g(b)$ (iii)

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Also, $h(x) = \begin{cases} f(x), & x \in [a, b) \\ f(b) = g(b), & x = b \\ g(x), & x \in (b, c] \end{cases}$

From (i) and (ii), we can conclude that $h(x)$ is continuous in $[a, b) \cup (b, c]$.

Also, $f(b^-) = f(b)$; $g(b^+) = g(b)$

$\therefore$ $h(b^-) = f(b^-) = f(b) = g(b) = g(b^+) = h(b^+)$

Obviously, $g(b^-)$ and $f(b^+)$ are undefined.

$h(b^-) = f(b^-) = f(b) = g(b) = g(b^+)$

and $h(b^+) = g(b^+) = g(b) = f(b) = f(b^-)$

Hence, $h(b^-) = h(b^+) = f(b) = g(b)$

Thus $h(x)$ has removable discontinuity at $x = b$.

17. (a, d)

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0}\left(1+\frac{ah+bh^3}{h^2}\right)^{1/h}$$

$$= \lim_{h\to 0} e^{\frac{1}{h}\ln\left(1+\frac{ah+bh^3}{h^2}\right)}$$

For limit to exist, we must have

$$\lim_{h\to 0}\frac{ah+bh^3}{h^2} = 0$$

$\Rightarrow$ $\lim_{h\to 0}\frac{a+bh^2}{h} = 0$

$\therefore$ $a = 0$

So, we have

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0}(1+bh)^{1/h}$$

$$= \lim_{h\to 0}((1+bh)^{1/bh)b} = e^b$$

For $f(x)$ to be continuous at $x = 0$, we must have

$$\lim_{x\to 0^+} f(x) = f(0)$$

$\Rightarrow$ $e^b = 3$

$\therefore$ $b = \log_e 3$

DPP 3.2

Single Correct Answer Type

1. (b) $\lim_{x\to 0} f(x) = \frac{1}{3}$

$\Rightarrow$ $\lim_{x\to 0}\frac{x\cdot\alpha\cot x+\beta}{x^2} = \frac{1}{3}$

$\Rightarrow$ $\lim_{x\to 0}\frac{x\alpha+\beta\tan x}{x^2\cdot\tan x} = \frac{1}{3}$

$\Rightarrow$ $\lim_{x\to 0}\frac{\alpha x+\beta\left(x+\frac{x^3}{3}+\ldots^{\infty}\right)}{x^3\left(\frac{\tan x}{x}\right)} = \frac{1}{3}$

$\Rightarrow$ $\lim_{x\to 0}\frac{(\alpha+\beta)x+\left(\frac{\beta}{3}\right)x^3+\ldots^{\infty}}{x^3} = \frac{1}{3}$

So $\alpha+\beta = 0$

Also, $\beta = 1 \Rightarrow \alpha = -1$

$\Rightarrow$ $\alpha^2+\beta^2 = 2$

2. (c) $f(x) = \begin{cases} \frac{2}{1+x^2}, & x \text{ is irrational} \\ b, & x \text{ is rational} \end{cases}$ has exactly two points of continuity then equation $\frac{2}{1+x^2} = b$ must have two distinct roots.

Now $\frac{2}{1+x^2} \in (0, 2]$

$\Rightarrow$ $b \in (0, 2]$

3. (b) $f(a^-) = \lim_{h\to 0}\frac{\left[\cos\left(\frac{\pi}{2a}(a-h)\right)\right]}{h} = \lim_{h\to 0}\frac{\left[\sin\frac{\pi h}{2a}\right]}{h} = 0;$

$f(a^+) = \lim_{h\to 0}\left(-\frac{\sin h}{2}\right)\tan\left[\frac{\pi}{2}+\frac{\pi h}{2}\right] = 0;$

Hence $f(a^+) = f(a^-)$.

Thus, $f(x)$ has removable discontinuity at $x = a$.

4. (c) When $x \in I$,

$[\cot x] = \cot x$

$\Rightarrow$ $\tan x\,[\cot x] = 1$

$\Rightarrow$ $[\tan x\,[\cot x]] = 1$

When $x \notin I$,

$[\cot x] < \cot x$

$\Rightarrow$ $0 < \tan x\,[\cot x] < 1$

$\Rightarrow$ $[\tan x\,[\cot x]] = 0$

$\Rightarrow$ $f(x) = [\tan x\,[\cot x]] = \begin{cases} 1, & \cot x \in I \\ 0, & \cot x \notin I \end{cases}$

So, $f(x)$ is discontinuous when $\cot x \in I$

Now $\frac{\pi}{12} \le x < \frac{\pi}{2}$

$\Rightarrow$ $0 < \cot x \le 2+\sqrt{3}$

Hence, number of points of discontinuity are $x = \cot^{-1}3, \cot^{-1}2$ and $\cot^{-1}1 = \frac{\pi}{4}$.

Thus three points of discontinuity.

5. (a) $f(x) = \begin{cases} \text{rational} & \text{if } x \notin Q \text{ in } [a, b] \\ \text{irrational} & \text{if } x \in Q \text{ in } [a, b] \end{cases}$

Let $c \in Q$

$\therefore$ $f(c) =$ irrational

$\lim_{x\to c} f(x) = \lim_{h\to 0} f(c+h)$ = rational or irrational

$\therefore f(x)$ is discontinuous for $x = c$, hence for all x in $[a, b]$

6. (a) $f(x) = [x]\,(\sin kx)^p$

$(\sin kx)^p$ is continuous function $\forall\, x \in R$, $k \in R$ and $p > 0$. $[x]$ is discontinuous at $x \in I$

For $k = n\pi$, $n \in I$

$f(x) = [x]\,(\sin(n\pi x))^p$

$\lim_{x\to a} f(x) = 0$, $a \in I$

and $f(a) = 0$

So, $f(x)$ becomes continuous for all $x \in R$

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7. (d) Clearly, $[2x-1]$ is discontinuous at three points $x=\frac{-5}{2},\frac{-3}{2}$ and -2

$f(x)$ may be continuous if $g(x)=ax^3+x^2+1=0$ at $x=\frac{-5}{2},\frac{-3}{2}$ or -2

$g(x)$ can be zero at atleast one point

$\therefore$ minimum number of points of discontinuity = 2.

8. (d) Domain of the function is $[-1, 1]$

$[\sin^{-1}x]$ is discontinuous at $x=-\sin 1, 0, \sin 1$

$[x]$ is discontinuous at $x=0, 1$

Now $f(0^+)=[0^+]-[0^+]=0-0=0$

$f(0^-)=[0^-]-[0^-]=-1-(-1)=0$

Hence $f(x)$ is continuous at $x=0$.

Therefore points of discontinuity are $-\sin 1$, $\sin 1$ and 1.

9. (b)
$$\lim_{x\to 1^+} g(x)=\lim_{x\to 1^+}\left\{\lim_{m\to\infty}\frac{x^m f(x)+h(x)+3}{2x^m+4x+1}\right\}$$
$$=\lim_{m\to\infty}\lim_{x\to 1^+}\left\{\frac{x^m f(x)+h(x)+3}{2x^m+4x+1}\right\}$$
$$=\lim_{m\to\infty}\lim_{x\to 1^+}\left\{\frac{f(x)+\frac{h(x)+3}{x^m}}{2+\frac{4x+1}{x^m}}\right\}$$
$$=\frac{f(1)}{2}$$

$$\lim_{x\to 1^-} g(x)=\lim_{x\to 1^-}\left\{\lim_{m\to\infty}\frac{x^m f(x)+h(x)+3}{2x^m+4x+1}\right\}$$
$$=\lim_{m\to\infty}\lim_{x\to 1^-}\left\{\frac{x^m f(x)+h(x)+3}{2x^m+4x+1}\right\}$$
$$=\frac{h(1)+3}{5}$$

$g(x)$ is continuous at $x=1$

$\therefore \quad \frac{f(1)}{2}=\frac{h(1)+3}{5}$

$\Rightarrow \quad 5f(1)-2h(1)=6$

10. (b) Given, $f(x)=([2x]+\{2x\})([2x]-\{2x\})=4x-4x\{2x\}$

$2x\in(-4, 4)$

Hence $f(x)$ is discontinuous when $2x=-3,-2,-1,1,2,3$.

At $x=0$, $f(x)$ is continuous.

11. (d) $h(x)=f(x)+g(x)$

$$=\begin{cases}-1, & -\infty<x<1\\ a+4-2x, & 1\le x<2\\ a-b-1+x, & 2\le x<\infty\end{cases}$$

$\therefore$ We must have either $a=-3$, $b\neq 1$ or $b=1$, $a\neq -3$

12. (c) $f(x)=\frac{x^3}{8}-\sin\pi x+4$

$f(-4)=-4, f(4)=12$

$\Rightarrow f(x)$ can take value 10 as $f(x)$ is continuous function

$\frac{x^3}{8}+4\le 12$ and $|\sin\pi x|\le 1\ \forall\ x\in[-4,4]$

$\therefore f(x)$ cannot take the value 18

13. (c) Given $f(2x)-f(x)=x$,

Replacing x by $x/2$, we have $f(x)-f\left(\frac{x}{2}\right)=\frac{x}{2}$

Replacing x by $\frac{x}{2}$ repeatedly and adding, we get

$$\left(f(x)-f\left(\frac{x}{2}\right)\right)+\left(f\left(\frac{x}{2}\right)-f\left(\frac{x}{4}\right)\right)+\cdots+\left(f\left(\frac{x}{2^{n-1}}\right)-f\left(\frac{x}{2^n}\right)\right)$$
$$=\frac{x}{2}+\frac{x}{2^2}+\frac{x}{2^3}+\cdots+\frac{x}{2^n}$$

$$\Rightarrow \quad f(x)-f\left(\frac{x}{2^n}\right)=\frac{x}{2}+\frac{x}{2^2}+\frac{x}{2^3}+\cdots+\frac{x}{2^n}$$

$$\therefore \quad \lim_{n\to\infty}\left(f(x)-f\left(\frac{x}{2^n}\right)\right)=\frac{\frac{x}{2}}{1-\frac{1}{2}}$$

$\Rightarrow \quad f(x)-f(0)=x$

$\left(\because f(x)\text{ is continuous} \Rightarrow \lim_{n\to\infty} f\left(\frac{x}{2^n}\right)=f(0)=1\right)$

$\Rightarrow \quad f(x)=x+f(0)=x+1$

$\Rightarrow \quad f(3)=4$

Multiple Correct Answers Type

14. (b, d)
$$\lim_{x\to 0^+}\left(3-\left[\cot^{-1}\left(\frac{2x^3-3}{x^2}\right)\right]\right)=3-[\cot^{-1}(-\infty)]=0$$
$$\lim_{x\to 0^-}\{x^2\}\cos(e^{1/x})=0\times(\text{value between } -1 \text{ and } 1)$$
$$=0$$

15. (a, b, c, d)
$$f(1^+)=\lim_{x\to 1^+}\left(x\left[\frac{1}{x}\right]+x[x]\right)$$
$$=\lim_{x\to 1^+}(x(0)+x(1))$$
$$=1$$

$$f(1^-)=\lim_{x\to 1^-}\left(x\left[\frac{1}{x}\right]+x[x]\right)$$
$$=\lim_{x\to 1^-}(x(1)+x(0))$$
$$=1$$

$$f(2^+)=\lim_{x\to 2^+}\left(x\left[\frac{1}{x}\right]+x[x]\right)$$
$$=\lim_{x\to 2^+}(x(0)+x(2))$$
$$=4$$

$$f(2^-)=\lim_{x\to 2^-}\left(x\left[\frac{1}{x}\right]+x[x]\right)$$
$$=\lim_{x\to 2^-}(x(0)+x(1))$$
$$=2$$

Obviously $f(x)$ is discontinuous at all positive integers but at $x=1$ it has removable discontinuity.

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16. (a, b, c)

$$f(x)=\lim_{n\to\infty}\frac{ax^2+bx+c+e^{nx}}{1+c\cdot e^{nx}}$$

$$=\begin{cases}\lim\limits_{n\to\infty}\dfrac{ax^2+bx+c+(e^x)^n}{1+c\cdot(e^x)^n}; & x<0\\ \lim\limits_{n\to\infty}\dfrac{ax^2+bx+c+(e^x)^n}{1+c\cdot(e^x)^n}; & x=0\\ \lim\limits_{n\to\infty}\dfrac{\dfrac{ax^2+bx+c}{(e^x)^n}+1}{\dfrac{1}{(e^x)^n}+c}; & x>0\end{cases}$$

$$=\begin{cases}ax^2+bx+c; & x<0\\ 1; & x=0\\ \dfrac{1}{c}; & x>0\end{cases}$$

since $f(x)$ is continuous function $\forall\, x\in R$

$\therefore \quad \lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^-}f(x)=f(0)$

$\Rightarrow \quad \lim\limits_{x\to0^+}\left(\dfrac{1}{c}\right)=\lim\limits_{x\to0^-}(ax^2+bx+c)=1$

$\Rightarrow \quad \dfrac{1}{c}=1=c\Rightarrow c=1$

$\therefore \quad c=1,\ a,\ b\in R$

DPP 3.3

Single Correct Answer Type

1. (c) $f(x)=\begin{cases}3x^2-4\sqrt{x}+1 & \text{for } x<1\\ ax+b & \text{for } x\geq1\end{cases}$

for continuity at $x=1$, $f(1^+)=f(1^-)$

$\Rightarrow \quad a+b=3-4+1$

$\Rightarrow \quad a+b=0$

for differentiability at $x=1$,

$f'(1^+)=f'(1^-)$

$\Rightarrow \quad a=6-2=4$

$\Rightarrow \quad b=-4$

2. (a) $f(x)=\begin{cases}[x]+\sqrt{\{x\}} & x<1\\ \dfrac{1}{[x]+\{x\}^2} & x\geq1\end{cases}$

Consider the function $f(x)$ in the interval $(0, 2)$.

$$f(x)=\begin{cases}\sqrt{x} & 0<x<1\\ \dfrac{1}{1+(x-1)^2} & 1\leq x<2\end{cases}$$

$f(1)=1$ and $\lim\limits_{x\to1}f(x)=1$

$\therefore f(x)$ is continuous at $x=1$

$$f'(1^+)=\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{\frac{1}{1+h^2}-1}{h}$$

$$=\lim_{h\to0^+}\frac{-h}{1+h^2}=0$$

$$f'(1^-)=\lim_{h\to0^+}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to0^+}\frac{\sqrt{1-h}-1}{-h}$$

$$=\lim_{h\to0^+}\frac{1}{\sqrt{1-h}+1}=\frac{1}{2}$$

$\therefore f(x)$ is not differentiable at $x=1$

3. (b) Let $f'(0^+)=\lim\limits_{h\to0}\dfrac{f(h)-f(0)}{h}=k$ (say)

$\therefore \quad f'(0^-)=\lim\limits_{h\to0}\dfrac{f(0)-f(0-h)}{h}$

$$=\lim_{h\to0}\frac{f(0)-f(h)}{h}=-k.$$

Since $f'(0^+)\neq f'(0^-)$, but both are finite, we can say that $f(x)$ is continuous at $x=0$ but not differentiable at $x=0$.

4. (c) $g(x)=f(10-2x)$

$\Rightarrow \quad g'(x)=-2f'(10-2x)$

$\Rightarrow \quad g'(2)=-2f'(6)$

Now $f(x)$ is differentiable.

Also $f'(x)<0$ on $(-4, 6)$

and $f'(x)>0$ on $(6, \infty)$

$\Rightarrow \quad f'(6)=0$

$\Rightarrow \quad g'(2)=0$

5. (a) Clearly $f(x)$ is continuous for all real x.

$$f(x)=x^2-|x^2-1|+2|x|-1|+2|x|-7$$

$$=\begin{cases}x^2-(x^2-1)+2(-x-1)-2x-7, & x<-1\\ x^2-(1-x^2)+2(x+1)-2x-7, & -1\leq x\leq0\\ x^2-(1-x^2)+2(1-x)+2x-7, & 0\leq x<1\\ x^2-(x^2-1)+2(x-1)+2x-7, & x>1\end{cases}$$

$$=\begin{cases}-4x-8, & x<-1\\ 2x^2-6, & -1\leq x<0\\ 2x^2-6, & 0\leq x<1\\ 4x-8, & x>1\end{cases}$$

$$f'(x)=\begin{cases}-4, & x<-1\\ 4x, & -1<x<0\\ 4x, & 0<x<1\\ 4, & x>1\end{cases}$$

Clearly, $f(x)$ is differentiable for all real x.

6. (d) $f(x)=|x-1|\cdot([x]-[-x])$

$[x]-[-x]$ is discontinuous at $x=1$

But $g(x)=|x-1|$ is continuous and $g(1)=0$

Hence $f(x)$ is continuous at $x=1$

$$\text{Also } f(x)=\begin{cases}(1-x)(0-(-1)), & x\in(0,1)\\ 0(1-(-1)), & x=1\\ (x-1)(1-(-2)), & x\in(1,2)\end{cases}$$

$$=\begin{cases}1-x, & x\in(0,1]\\ 3(x-1), & x\in(1,2)\end{cases}$$

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Clearly $f(x)$ is continuous at $x = 1$ and non-differentiable at $x = 1$ as $f'(1^-) = -1$ and $f'(1^+) = 3$

7. (b) $$f(x) = \begin{cases} 3 - \left|\cos x - \frac{1}{\sqrt{2}}\right|, & |\sin x| < \frac{1}{\sqrt{2}} \\ 2 + \left|\cos x + \frac{1}{\sqrt{2}}\right|, & |\sin x| \geq \frac{1}{\sqrt{2}} \end{cases}$$

$$= \begin{cases} 3 - \left|\cos x - \frac{1}{\sqrt{2}}\right|, & |\cos x| > \frac{1}{\sqrt{2}} \\ 2 + \left|\cos x + \frac{1}{\sqrt{2}}\right|, & |\cos x| \leq \frac{1}{\sqrt{2}} \end{cases}$$

$$= \begin{cases} 3 - \cos x + \frac{1}{\sqrt{2}}, & |\cos x| > \frac{1}{\sqrt{2}} \\ 2 + \cos x + \frac{1}{\sqrt{2}}, & |\cos x| \leq \frac{1}{\sqrt{2}} \end{cases}$$

Thus, $f(x)$ is discontinuous at $|\cos x| = \frac{1}{\sqrt{2}}$ or

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

8. (c) $f(x) = \begin{cases} [x] & x \notin I \\ x - 1 & x \in I \end{cases}$ (where, $[\cdot]$ denotes the greatest integer function) and $g(x) = \begin{cases} \sin x + \cos x, & x < 0 \\ 1, & x \geq 0 \end{cases}$. Then for $f(g(x))$ at $x = 0$

For continuity at $x = 0$, $f(g(0)) = f(1) = 1 - 1 = 0$

$f(g(0^+)) = f(1) = 1 - 1 = 0$

$f(g(0^-)) = f(\sin(0^-) + \cos(0^-)) = f((0^-) + (1^-)) = f(1^-) = [1^-] = 0$

Thus $f(x)$ is continuous at $x = 0$

For differentiability at $x = 0$,

$$f'(g(0^+)) = \lim_{h \to 0} \frac{f(g(h)) - f(g(0))}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

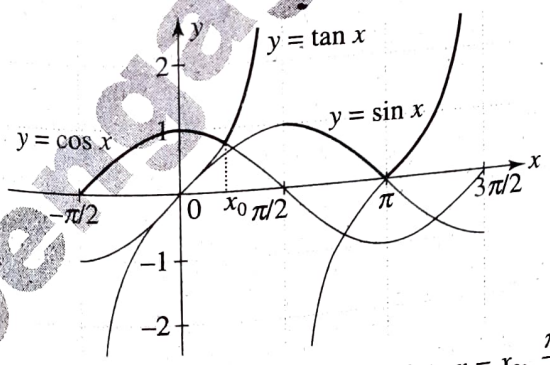
$$f'(g(0^-)) = \lim_{h \to 0} \frac{[1^-] - 0}{-h} = 0$$

$\therefore f(x)$ is differentiable at $x = 0$.

9. (d) $f(x) = \begin{cases} 2\sqrt{1 - x^2}, & x \leq 0 \\ 0, & x > 0 \end{cases}$

Clearly, $f(x)$ is discontinuous, hence non-differentiable at $x = 0$.

10. (b)



From the graph, there are three points $x = x_0, \frac{\pi}{2}, \pi$ of non-differentiability.

11. (c) $f(x) = \frac{1}{1 + \frac{1}{x}} = \frac{x}{x+1}$ is not differentiable at $x = 0, -1$.

Also, $g(x) = \frac{1}{1 + \frac{2}{f(x)}} = \frac{1}{1 + \frac{2(x+1)}{x}} = \frac{x}{3x+2}$

Thus, the points where $g(x)$ is not differentiable are $x = 0, -1, -2/3$.

12. (c) $$t_{r+1} = \frac{x}{(rx+1)\{(r+1)x+1\}}$$

$$= \frac{(r+1)x + 1 - (rx+1)}{(rx+1)[(r+1)x+1]}$$

$$= \frac{1}{(rx+1)} - \frac{1}{(r+1)x+1}$$

$$\therefore \quad S_n = \sum_{r=0}^{n-1} t_{r+1} = 1 - \frac{1}{nx+1}$$

$$\therefore \quad f(x) = 1 - \frac{1}{nx+1}, \; x \neq 0$$

$f(0) = 0$ and

$$\therefore \quad \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{nx+1}\right) = 1$$

$$\therefore \quad f(0^+) = 1$$

Thus, $f(x)$ is neither continuous nor differentiable at $x = 0$.

13. (b) $$f(x) = \begin{cases} \lim_{n \to \infty} \dfrac{ax(x-1)\left(\cot\frac{\pi x}{4}\right)^n + (px^2 + 2)}{\left(\cot\frac{\pi x}{4}\right)^n + 1}, & \\ 0, & x = 1 \end{cases}$$

$x \in (0, 1) \cup (1, 2)$

$$\therefore \quad f(x) = \begin{cases} ax(x-1), & 0 < x < 1 \\ 0, & x = 1 \\ px^2 + 2, & 1 < x < 2 \end{cases}$$

$\therefore f(x)$ must be continuous at $x = 1$

$\therefore$ L.H.L. = R.H.L = $f(1)$

$\Rightarrow \quad p + 2 = 0 \Rightarrow p = -2$

Also $f'(x) = \begin{cases} a(2x-1), & 0 < x < 1 \\ -4x, & x > 1 \end{cases}$

$\because$ $f(x)$ is differentiable at $x = 1$, so

L.H.D. = R.H.D. $\Rightarrow a = -4$

Hence $|a + p| = 6$

Multiple Correct Answers Type

14. (a, b)

$$g(x) = \begin{cases} \frac{f(x)}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$$

$$\lim_{x \to 0} \frac{f(x)}{x} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{f'(x)}{1}$$

$$= f'(0) = 0$$

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Thus $g(x)$ is continuous at $x = 0$

$$g'(0^+) = \lim_{h\to 0} \frac{\frac{f(h)}{h} - g(0)}{h}$$

$$= \lim_{h\to 0} \frac{f''(h)}{2}$$

$$= \frac{f''(0)}{2}$$

Similarly

$$g'(0^-) = \frac{f''(0)}{2}$$

15. (b, c)

$\because$ $\lim_{x\to 0} \frac{f(x) - f(k\alpha)}{x} = \alpha$

$\Rightarrow$ $\lim_{x\to 0} \frac{f(x) - f(0) + f(0) - (kx)}{x} = \alpha$

$\Rightarrow$ $\lim_{x\to 0} \left(\frac{f(x) - f(0)}{x} - \frac{f(kx) - f(0)}{x} \right) = \alpha$

$\Rightarrow$ $\left(\lim_{x\to 0} \frac{f(x) - f(0)}{x} \right) - \left(\lim_{x\to 0} \frac{f(kx) - f(0)}{kx} \right) k = \alpha$

$\Rightarrow$ $\begin{cases} \lim_{x\to 0^-} \frac{f(x) - f(0)}{x} - \lim_{x\to 0^-} \frac{f(kx) - f(0)}{kx} \cdot k = \alpha \\ \lim_{x\to 0^+} \frac{f(x) - f(0)}{x} - \lim_{x\to 0^+} \frac{f(kx) - f(0)}{kx} \cdot k = \alpha \end{cases}$

$$= \begin{cases} f'(0^-) - kf'(0^-) = \alpha \\ f'(0^+) - kf'(0^+) = \alpha \end{cases}$$

$$= \begin{cases} (1-k) f'(0^-) = \alpha \\ (1-k) f'(0^+) = \alpha \end{cases}$$

$$= \begin{cases} f'(0^-) = \frac{\alpha}{1-k} \\ f'(0^+) = \frac{\alpha}{1-k} \end{cases}$$

$\therefore$ $f'(0) = f'(0^-) = f'(0^+) = \frac{\alpha}{1-k}$

16. (c)

$$f(x) = x^{1/3} (x-2)^{2/3}$$

$\therefore$ $f'(x) = x^{1/3} \cdot \frac{2}{3} (x-2)^{-1/3} + (x-2)^{2/3} \cdot \frac{1}{3} x^{-2/3}$

$$= \frac{1}{3} x^{-2/3} (x-2)^{-1/3} (3x - 2)$$

$\therefore$ f' is not defined at $x = 0$ and at $x = 2$

Comprehension Type

For Questions 17 and 18

Given two functions: $f(x) = \begin{cases} ax^2 + b; & 0 \le x \le 1 \\ bx + 2b; & 1 < x \le 3 \\ (a-1)x + 2c - 3; & 3 < x \le 4 \end{cases}$

and $g(x) = \begin{cases} cx + d; & 0 \le x \le 2 \\ ax + 3 - c; & 2 < x < 3 \\ x^2 + b + 1; & 3 \le x \le 4 \end{cases}$

Condition for continuity of $f(x)$: $f(1^-) = f(1) = f(1^+)$ and $f(3^-) = f(3) = f(3^+)$

$\Rightarrow$ $a + b = 3b$ and $5b = 3a + 2c - 6$

$\Rightarrow$ $a = 2b$ and $c = 3 - \frac{b}{2}$

condition for continuity of $g(x)$:

$g(2) = g(2^-) = g(2^+)$ and $g(3^-) = g(3) = g(3^+)$

$\Rightarrow$ $2c + d = 2a + 3 - c$ and $3a + 3 - c = 10 + b$

$\Rightarrow$ $3c + d - 2a = 3$ and $b + c - 3a = -7$

Also $f'(x) = \begin{cases} 2ax; & 0 < x < 1 \\ b; & 1 < x < 3 \\ a - 1; & 3 < x < 4 \end{cases}$ and $g'(x) = \begin{cases} c; & 0 < x < 2 \\ a; & 2 < x < 3 \\ 2x; & 3 < x < 4 \end{cases}$

17. (a) f is differentiable at $x = 1$ and $g(x)$ is continues at $x = 3$.

i.e. $a = 2b$ and $2a = b$

Also $3a + 3 - c = 10 + b$

$\Rightarrow$ $a = b = 0$ and $c = -7$

$x^2 - 7\alpha x + 49k\,(1 + \alpha) = 0$ has real and distinct roots for $\forall \alpha \in R$.

$\Rightarrow$ $49\alpha^2 - 4(49k)\,(1 + \alpha) > 0\ \forall \alpha \in R$

$\Rightarrow$ $\alpha^2 - 4k\alpha - 4k > 0\ \forall \alpha \in R$

$\Rightarrow$ $16k^2 + 16k < 0$

$\Rightarrow$ $k(k + 1) < 0$

$k \in (-1, 0)$

18. (c) $\text{LHL} = \lim_{h\to 0} \frac{f(2-h)}{|g(2-h)| + 1} = \lim_{h\to 0} \frac{b(4-h)}{|c(2-h) + d| + 1}$

$$= \frac{4b}{|2c + d| + 1}$$

$\text{RHL} = \lim_{h\to 0} \frac{f(2+h)}{|g(2+h)| + 1} = \lim_{h\to 0} \frac{b(4+h)}{|(a(2+h) + 3 - c)| + 1}$

$$= \frac{4b}{|2a + 3 - c| + 1}$$

$\because$ f is differentiable at $x = 1$ i.e., $a = b = 0$

So, LHL = RHL = 0 $\because$ (it is given that limit exists)

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CHAPTER 4

DPP 4.1

Single Correct Answer Type

1. (b) $f'(7^+) = \lim_{h\to 0} \frac{f(7+h)-f(7)}{h}$

$= \lim_{h\to 0} \frac{[7+h]\tan \pi(7+h) - [7]\tan 7\pi}{h}$

$= \lim_{h\to 0} \frac{7\tan \pi(7+h)}{h}$

$= 7\pi \lim_{h\to 0} \frac{\tan \pi h}{\pi h}$

$= 7\pi$

2. (b) From the given functional relations, we can consider

$f(x) = \sin x$ and $g(x) = \cos x$

$\therefore \quad g'(x) = -\sin x$

$\therefore \quad g'(0) = 0$

3. (b) We have $xe^{xy} - y = \sin^2 x$

When $x = 0$, $y = 0$

Differentiating w.r.t. x, we get

$1\times e^{xy} + x\times e^{xy}(y + xy') - y' = 2\sin x \cos x$

$\therefore \quad \frac{dy}{dx} = -\left(\frac{x\cdot ye^{xy} + e^{xy} - 2\sin x\cos x}{x^2\cdot e^{xy} - 1}\right)$

$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(0,0)} = 1$

4. (c) $y = fgh$

$\frac{dy}{dx} = f'gh + fg'h + fgh'$

$= \frac{1}{2}(2f'gh + 2fg'h + 2fgh')$

$= \frac{1}{2}(h(f'g + g'f) + g(f'h + fh') + f(g'h + gh'))$

$= \frac{1}{2}[h.(fg)' + g.(fh)' + f.(gh)']$

$\therefore \quad (fgh)'(0) = \frac{1}{2}[h(0)(fg)'(0) + g(0)(fh)'(0) + f(0)(gh)'(0)]$

$= \frac{1}{2}(3\times 6 + 2\times 5 + 1\times 4)$

$= \frac{1}{2}(18 + 10 + 4) = \frac{32}{2} = 16$

5. (b) $y(x) = f(e^x)\, e^{f(x)}$

$\therefore \quad y'(x) = f(e^x)\cdot e^{f(x)}.f'(x) + e^{f(x)} f'(e^x)\cdot e^x$

$\therefore \quad y'(0) = f(1)\cdot e^{f(0)}\cdot f'(0) + e^{f(0)} f'(1)$

$= 0 + e^0.2 = 2$

6. (a) Let $x = \cos\theta$, $\therefore \theta = \cos^{-1}x$

$\therefore \quad y = \cos\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right) - 2\cos^{-1}\sqrt{\frac{1-x}{2}}$

$= \cos\left(2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right) - 2\cos^{-1}\left(\sin\frac{\theta}{2}\right)$

$= \cos\theta - 2\left(\frac{\pi}{2} - \sin^{-1}\left(\sin\frac{\theta}{2}\right)\right)$

$= \cos\theta - \pi + \theta$

$\therefore \quad y = x - \pi + \cos^{-1}x$

$\therefore \quad \frac{dy}{dx} = 1 - \frac{1}{\sqrt{1-x^2}}$

7. (b) $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log(x + \sqrt{(x^2+1)})$

$\Rightarrow \quad y' = x + \frac{1}{2}\sqrt{x^2+1} + \frac{1}{2}x\frac{x}{\sqrt{x^2+1}}$

$+ \frac{1}{2(x+\sqrt{x^2+1})}\times\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$

$= x + \frac{(x^2+1+x^2)}{2\sqrt{x^2+1}} + \frac{1}{2\sqrt{x^2+1}}$

$= x + \frac{2x^2+1}{2\sqrt{x^2+1}} + \frac{1}{2\sqrt{x^2+1}}$

$= x + \frac{2x^2+1+1}{2\sqrt{x^2+1}}$

$\Rightarrow \quad y' = x + \sqrt{x^2+1}$

$\Rightarrow \quad xy' + \log y' = x(x + \sqrt{x^2+1}) + \log(x + \sqrt{x^2+1})$

$= x^2 + x\sqrt{x^2+1} + \log(x + \sqrt{x^2+1})$

$= 2y$

8. (c) We have $g(x) = f(x)\sin x$ (1)

On differentiating equation (1) w.r.t. x, we get

$g'(x) = f(x)\cos x + f'(x)\sin x$ (2)

Again differentiating equation (2) w.r.t. x, we get

$g''(x) = f(x)(-\sin x) + f'(x)\cos x + f'(x)\cos x + f''(x)\sin x$ (3)

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$\Rightarrow \quad g''(-\pi) = 2f'(-\pi)\cos(-\pi) = 2 \times 1 \times -1 = -2$

Hence $g''(-\pi) = -2$

9. (d) $f(x) = \dfrac{\log_e(\log_e x)}{\log_e x}$

$$\therefore \quad f'(x) = \frac{\frac{1}{\log_e x} \times \frac{1}{x} \times \log_e(x) - \log_e(\log_e(x)) \times \frac{1}{x}}{[\log_e(x)]^2}$$

$$\therefore \quad f'(e) = \frac{\frac{1}{e} - 0}{1} = 1/e$$

10. (c) $f(x+1) = x + f(x)$

$\Rightarrow \quad e^{f(x+1)} = e^x + f(x) = e^x \times e^{f(x)}$

$\Rightarrow \quad g(x+1) = e^x \times g(x)$

$\Rightarrow \quad \ln(g(x+1)) = x + \ln(g(x))$

$$\Rightarrow \quad \frac{g'(x+1)}{g(x+1)} - \frac{g'(x)}{g(x)} = 1$$

$$\Rightarrow \quad \frac{g'\left(\frac{1}{2}+1\right)}{g\left(\frac{1}{2}+1\right)} - \frac{g'\left(\frac{1}{2}\right)}{g\left(\frac{1}{2}\right)} = 1$$

$$\frac{g'\left(\frac{1}{2}+2\right)}{g\left(\frac{1}{2}+2\right)} - \frac{g'\left(1+\frac{1}{2}\right)}{g\left(1+\frac{1}{2}\right)} = 1$$

$$\vdots \qquad \vdots$$

$$\frac{g'\left(n+\frac{1}{2}\right)}{g\left(n+\frac{1}{2}\right)} - \frac{g'\left(n-\frac{1}{2}\right)}{g\left(n-\frac{1}{2}\right)} = 1$$

Adding $\quad \dfrac{g'\left(n+\frac{1}{2}\right)}{g\left(n+\frac{1}{2}\right)} - \dfrac{g'\left(\frac{1}{2}\right)}{g\left(\frac{1}{2}\right)} = n$.

11. (b) $y = \cos^{-1}[x\sqrt{x} - \sqrt{1-(\sqrt{x})^2}\sqrt{1-x^2}]$

$= \cos^{-1} x + \cos^{-1}\sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}$$

Multiple Correct Answers Type

12. (b, d)

Put $x = \tan\theta$

$$\therefore \quad \tan^{-1} x = \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \quad y = \cos^{-1}\sqrt{\frac{1+|\sec\theta|}{2|\sec\theta|}}$$

$$= \cos^{-1}\sqrt{\frac{\cos\theta+1}{2}}$$

$$= \cos^{-1}(\cos\theta/2)$$

$$= \begin{cases} -\theta/2, & -\pi/2 < \theta \le 0 \\ \theta/2, & 0 < \theta < \pi/2 \end{cases}$$

$$= \begin{cases} -\frac{1}{2}\tan^{-1} x, & x \le 0 \\ \frac{1}{2}\tan^{-1} x, & x > 0 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} -\frac{1}{2(1+x^2)}, & x \le 0 \\ \frac{1}{2(1+x^2)}, & x > 0 \end{cases}$$

13. (a, d)

$h'(x) = f(x)$

$h''(x) = f'(x)$

$h(1) = 0, f(1) = f'(1) = h'(1) = h''(1) = 1 = g(1)$

'g' is inverse of 'f'

$\therefore \quad f(g(x)) = x$

$\Rightarrow \quad f'(g(x)).\ g'(x) = 1$

$\Rightarrow \quad f'(g(1)).\ g'(1) = 1$

$\Rightarrow \quad f'(1)\ .\ g'(1) = 1$

$\Rightarrow \quad g'(1) = 1$

$G(x) = x^2 g(x) - xh(g(x))$

$G'(x) = 2x\ g(x) + x^2 g'(x) - h(g(x)) - xh'(g(x)).\ g'(x)$

$= 2xg(x) + x^2 g'(x) - h(g(x)) - x^2 g'(x)$

$(\because\ h'(x) = f(x) \quad \therefore\ h'(g(x)) = f(g(x)) = x)$

$= 2xg(x) - h(g(x))$

$G''(x) = 2g(x) + 2xg'(x) - h'(g(x))\ .\ g'(x)$

$= 2g(x) + 2xg'(x) - f(g(x))\ .\ g'(x)$

$= 2g(x) + xg'(x)$

$G'(1) = 2g(1) - h(g(1)) = 2g(1) - h(1) = 2 - 0 = 2$

$G''(1) = 2g(1) + g'(1) = 3$

DPP 4.2

Single Correct Answer Type

1. (c) $\dfrac{dy}{dx} = 2x + 2t \cdot \dfrac{dt}{dx}$

Also, $t = \dfrac{x}{1+x^2}$

$$\therefore \quad \frac{dt}{dx} = \frac{1-x^2}{(1+x^2)^2}$$

Now putting $x = 2$, we get

$$\frac{dt}{dx} = \frac{-3}{25}$$

$$\therefore \quad \frac{dy}{dx} = 2(2) + 2\left(\frac{2}{5}\right)\cdot\frac{-3}{25} = 4 - \frac{12}{125} = \frac{488}{125}$$

2. (a) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-\frac{(3+2t)}{t^3}}{-\frac{(3+2t)}{t^4}} = t$

Since $f(x)\left(\dfrac{dy}{dx}\right)^3 = 1 + \dfrac{dy}{dx}$

$\Rightarrow \quad f(x)\, t^3 = 1 + t$

$\Rightarrow \quad f(x) = \dfrac{1+t}{t^3} = x$

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3. (c) $y = x^3 - 8x + 7$ and $x = f(t)$

$$\frac{dy}{dt} = 2 \text{ and } x = 3 \text{ at } t = 0$$

$$\because \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{dy/dt}{dy/dx}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2}{3x^2 - 8}$$

$$\because \quad \text{at } t = 0,\ x = 3$$

$$\therefore \quad \frac{dx}{dt} \text{ (at } t = 0) = \frac{2}{19}$$

4. (c) $$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{8\sec^8\theta\tan\theta + 8\cos^7\theta\sin\theta}{\sec\theta\tan\theta + \sin\theta}$$

$$= \frac{8\tan\theta(\sec^8\theta + \cos^8\theta)}{\tan\theta(\sec\theta + \cos\theta)}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{64(\sec^8\theta + \cos^8\theta)^2}{(\sec\theta + \cos\theta)^2}$$

$$= \frac{64[(\sec^8\theta - \cos^8\theta)^2 + 4]}{[(\sec\theta - \cos\theta)^2 + 4]}$$

$$= \frac{64(y^2 + 4)}{(x^2 + 4)}$$

$$\therefore \quad \left(\frac{x^2 + 4}{y^2 + 4}\right)\left(\frac{dy}{dx}\right)^2 = 64$$

5. (b) We have,

$$x = 2t - |t|,\ y = t^3 + t^2|t|$$

$$\Rightarrow \quad x = 3t,\ y = 0 \text{ when } t < 0$$

$$x = t,\ y = 2t^3 \text{ when } t \geq 0$$

Eliminating the parameter t, we get

$$y = \begin{cases} 0, & x < 0 \\ 2x^3, & x \geq 0 \end{cases}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \begin{cases} 0, & x < 0 \\ 6x^2, & x \geq 0 \end{cases}$$

Hence, the function is differentiable everywhere and its derivative at $x = 0$ $(t = 0)$ is 0.

6. (c) $y = \tan^{-1}\dfrac{u}{\sqrt{1-u^2}}$

$$\Rightarrow \quad y = \sin^{-1}u$$

Also, $x = \sec^{-1}\dfrac{1}{2u^2 - 1}$

$$\Rightarrow \quad x = \cos^{-1}(2u^2 - 1)$$

$$y = \sin^{-1}u$$

$$\therefore \quad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$= \cos^{-1}(2u^2 - 1)$$

Also, $$\frac{dx}{du} = \frac{-1}{\sqrt{1-(2u^2-1)^2}}(4u)$$

$$= -\frac{4u}{\sqrt{1 - 4u^4 - 1 + 4u^2}}$$

$$= -\frac{4u}{\sqrt{4u^2(1-u^2)}} = -\frac{2}{\sqrt{1-u^2}}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}$$

$$\Rightarrow \quad 2\left(\frac{dy}{dx}\right) + 1 = 0$$

7. (d) Let $u = \sin^{-1}\left(\dfrac{4}{5}\sin 2x + \dfrac{3}{5}\cos 2x\right) = \sin^{-1}\sin(\alpha + 2x)$,

where $\sin\alpha = \dfrac{3}{5}, \cos\alpha = \dfrac{4}{5}$

i.e., $u = \alpha + 2x$,

Similarly, $y = \cos^{-1}\left(\dfrac{5\cos x - 4\sin x}{\sqrt{41}}\right)$

$$= \cos^{-1}\cos(\beta + x),$$

Here, $\cos\beta = \dfrac{5}{\sqrt{41}}$, $\sin\beta = \dfrac{4}{\sqrt{41}}$

i.e., $v = \beta + x$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2}{1} = 2.$$

8. (b) Taking log on both sides we get

$$n\log(x + y) = \log x + \log y$$

Differentiating w.r.t. x, we get

$$\frac{n}{x+y}\left[1 + \frac{dy}{dx}\right] = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left[\frac{x + y - nx}{ny - x - y}\right]$$

$$\Rightarrow \quad n = 2 \text{ for } \frac{dy}{dx} = \frac{y}{x}$$

9. (b) $y = 3e^2 - x$

$$\therefore \quad x^y = x^{3e^2 - x}$$

$$f(x) = x^{3e^2 - x}$$

$$\therefore \quad \log(f(x)) = (3e^2 - x)\log x$$

$$\therefore \quad \frac{1}{f(x)} \cdot f'(x) = \frac{3e^2 - x}{x} - \log x$$

Now $f'(x) = 0$

$$\Rightarrow \quad 3e^2 - x = x\log x$$

$$\Rightarrow \quad x = e^2$$

10. (a) $f(x) = \prod_{i=1}^{100}(x - i)^{i(101-i)}$

Taking log both sides, we get

$$\log(f(x)) = \sum_{i=1}^{100} i(101 - i)\log(x - i)$$

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Differentiating w.r.t. x, we get

$$\frac{1}{f(x)} \cdot f'(x) = \sum_{i=1}^{100} \frac{i(101-i)}{(x-i)}$$

$$\Rightarrow \quad \frac{f'(101)}{f(101)} = \sum_{i=1}^{100} i\left(\frac{101-i}{101-i}\right) = 5050$$

Multiple Correct Answers Type

11. (a, d)

$$\frac{dy}{dx} = \frac{2t}{1+3t^2} = \frac{1}{2}$$

$$\Rightarrow \quad 3t^2 - 4t + 1 = 0$$

12. (b, d)

$$y = x^{(\log x)^{\log(\log x)}}$$

$$\Rightarrow \quad \log y = (\log x)\,(\log x)^{\log(\log x)} \qquad (1)$$

Taking log on both sides, we get

$$\log(\log y) = \log(\log x) + \log(\log x)\log(\log x)$$

Differentiating w.r.t. x, we get

$$\frac{1}{\log y} \cdot \frac{1}{y}\frac{dy}{dx} = \frac{1}{x\log x} + \frac{2\log(\log x)}{\log x}\frac{1}{x}$$

$$= \frac{2\log(\log x)+1}{x\log x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} \cdot \frac{\log y}{\log x}(2\log(\log x)+1)$$

Substituting the value of $\log y$ from (1), we get

$$\frac{dy}{dx} = \frac{y}{x}(\log x)^{\log(\log x)}(2\log(\log x)+1)$$

DPP 4.3

Single Correct Answer Type

1. (c) We have $f(x^2) \cdot f''(x) = f'(x) \cdot f'(x^2)$ (i)

Differentiating w.r.t. x,

$$2xf'(x^2)\,f''(x) + f(x^2)\,f'''(x) = f''(x)\,f'(x^2) + 2xf''(x^2)\,f'(x)$$

Putting $x = 1$, we get

$$2f'(1)\,f''(1) + f(1)\,f'''(1) = f''(1) \cdot f'(1) + 2f''(1)f'(1)$$

$$\therefore \quad 2f'(1)\,f''(1) + 1 \times 8 = 3f''(1)\,f'(1)$$

$$\therefore \quad f'(1)\,f''(1) = 8 \qquad (ii)$$

Putting $x = 1$ in (i)

$$f(1)\,f''(1) = f'(1)\,f'(1)$$

$$\therefore \quad f''(1) = (f'(1))^2$$

So from (ii) $f'(1).\,(f'(1))^2 = 8$

$$\Rightarrow \quad f'(1) = 2 \text{ and } f''(1) = 4$$

$$\Rightarrow \quad f'(1) + f''(1) = 6$$

2. (a) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)$$

$$= \frac{\left(\frac{dx}{dt}\right)\frac{d}{dx}\left(\frac{dy}{dt}\right) - \left(\frac{dy}{dt}\right)\frac{d}{dx}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}$$

$$= \frac{\left(\frac{dx}{dt}\right)\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} - \left(\frac{dy}{dt}\right)\frac{d}{dt}\left(\frac{dx}{dt}\right)\frac{dt}{dx}}{\left(\frac{dx}{dt}\right)^2}$$

$$= \frac{\left(\frac{dx}{dt}\right)\frac{d^2y}{dt^2} - \left(\frac{dy}{dt}\right)\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

3. (a) $\sin x + \sin y = 1$

$$\Rightarrow \quad \cos x + \cos y\, y' = 0$$

$$\Rightarrow \quad -\sin x + \cos y\, y'' + (y')\,(-\sin y) = 0$$

$$\Rightarrow \quad y'' = \frac{y'\sin y + \sin x}{\cos y}$$

$$= \frac{\left(-\frac{\cos x}{\cos y}\right)^2 \sin y + \sin x}{\cos y}$$

$$= \frac{\sin x \cos^2 y + \cos^2 x \sin y}{\cos^3 y}$$

Putting $\sin x = t$, $\sin y = 1 - t$, we get

$$y'' = \frac{t^{-3/2}(1 - t + t^2)}{(2-t)^{3/2}}$$

$$\Rightarrow \quad \alpha = \frac{3}{2}$$

$$\Rightarrow \quad 2\alpha = 3$$

4. (c) $y = \left(\dfrac{\alpha x + \beta}{\gamma x + \delta}\right)$

$$\therefore \quad y_1 = \frac{\alpha\delta - \beta\gamma}{(\gamma x + \delta)^2}$$

$$y_2 = (\alpha\delta - \beta\gamma)\left[\frac{-2}{(\gamma x + \delta)^3}(\gamma)\right]$$

and $$y_3 = (\alpha\delta - \beta\gamma)\left[\frac{6}{(\gamma x + \delta)^4}(\gamma^2)\right]$$

$$\therefore \quad y_2^2 = \frac{2}{3}y_1 y_3$$

5. (b) We know that $\dfrac{d^2x}{dy^2} = \dfrac{-\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3} = \dfrac{-4}{8} = -\dfrac{1}{2}$

6. (c) $$\frac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \frac{16}{x-2} - \frac{1}{x-1}$$

$$\text{Third order derivative} = 0 + \frac{16(-3!)}{(x-2)^4} + \frac{3!}{(x-1)^4}$$

$$\therefore \quad -12k = -96$$

$$\therefore \quad k = 8$$

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7. (b) $\frac{y}{x} = \log x - \log(a+bx)$

$$\therefore \quad \frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{b}{b+ax} = \frac{a}{x(a+bx)}$$

$$\therefore \quad x\frac{dy}{dx} - y = \frac{ax}{a+bx} \qquad (1)$$

Differentiating again, we get

$$x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\therefore \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(\frac{xdy}{dx} - y\right)^2$$

8. (a) Now $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$

$$\therefore \quad \frac{d^2x}{dy^2} = \frac{1}{dy}\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{dx}\frac{1}{\left(\frac{dy}{dx}\right)} \times \frac{dx}{dy}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)^2} \times \frac{d^2y}{dx^2} \times \left(\frac{dx}{dy}\right)$$

$$\therefore \quad \frac{d^2x}{dy^2} = -\frac{1}{\left(\frac{dy}{dx}\right)^3} \times \frac{d^2y}{dx^2}$$

$$\therefore \quad \left(\frac{dy}{dx}\right) = \left(\frac{-\frac{d^2y}{dx^2}}{\left(\frac{d^2x}{dy^2}\right)}\right)^{1/3}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-\frac{d^2y}{dx^2}}{\frac{d^2x}{dy^2}}\right)^{2/3}$$

So, $$R^{2/3} = \frac{1 + \left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^{2/3}} = \frac{1}{\left(\frac{d^2y}{dx^2}\right)^{2/3}} + \frac{\left(\frac{d^2y}{dx^2}\right)^{2/3}}{\left(\frac{d^2y}{dx^2}\right)^{2/3}\left(\frac{d^2x}{dy^2}\right)^{2/3}}$$

$$= \frac{1}{\left(\frac{d^2y}{dx^2}\right)^{2/3}} + \frac{1}{\left(\frac{d^2x}{dy^2}\right)^{2/3}}$$

9. (c) We have $x = 2\cos t - \cos 2t$

and $y = 2\sin t - \sin 2t$

$$\therefore \quad \frac{dx}{dt} = -2\sin t + 2\sin 2t$$

$$= -2(\sin t - \sin 2t)$$

And $$\frac{dy}{dt} = 2\cos t - 2\cos 2t$$

$$= 2(\cos t - \cos 2t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos 2t - \cos t}{\sin t - \sin 2t}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{-(\cos 2t - \cos t)(\cos t - 2\cos 2t)}{(\sin t - \sin 2t)^2} \times \frac{dt}{dx}$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = \frac{1(1) - (-1)(+2)}{(1)^2} \times \frac{1}{-2(1)}$$

$$= \frac{3}{2}$$

10. (c) We have,

$$y^3 - y = 2x$$

Differentiating both sides w.r.t. x, we get

$$(3y^2 - 1)\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{2}{3y^2 - 1} \qquad (1)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-2 \cdot 6y\frac{dy}{dx}}{(3y^2-1)^2}$$

Using (1), we get

$$\frac{d^2y}{dx^2} = \frac{-24y}{(3y^2-1)^3} \qquad (2)$$

Now,

$$\left(x^2 - \frac{1}{27}\right)\frac{d^2y}{dx^2} + \frac{xdy}{dx}$$

$$= \left(x^2 - \frac{1}{27}\right)\left(\frac{-24y}{(3y^2-1)^3}\right) + \frac{2x}{(3y^2-1)}$$

$$= \left(\frac{y^2(y^2-1)^2}{4} - \frac{1}{27}\right)\left(\frac{-24y}{(3y^2-1)^3}\right) + \frac{y(y^2-1)}{3y^2-1}$$

$(\because y^3 - y = 2x)$

$$= \frac{\{27y^2(y^2-1)^2 - 4\}}{108}\frac{(-24y)}{(3y^2-1)^3} + \frac{y(y^2-1)}{3y^2-1}$$

$$= \frac{y}{9}\left\{\frac{-54y^2(y^2-1)^2 + 8}{(3y^2-1)^3} + \frac{9(y^2-1)}{3y^2-1}\right\}$$

$$= \frac{y}{9}\left\{\frac{-2(1+\alpha)(\alpha-2)^2 + 8}{\alpha^3}\right\} + \frac{3(\alpha-2)}{\alpha}$$

where $\alpha = 3y^2 - 1$

$$= \frac{y}{9}$$

Multiple Correct Answers Type

11. (a, b)

$$y = e^{-x}\cos x$$

$$\therefore \quad y_1 = -e^{-x}\cos x - e^{-x}\sin x$$

$$= -\sqrt{2}\,e^{-x}\cos\left(x - \frac{\pi}{4}\right)$$

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Similarly, we get

$$y_2 = (-\sqrt{2})^2 e^{-x}\cos\left(x-\frac{\pi}{2}\right)$$

$$y_3 = (-\sqrt{2})^3 e^{-x}\cos\left(x-\frac{3\pi}{4}\right)$$

$$y_4 = (-\sqrt{2})^4 e^{-x}\cos(x-\pi) = -4e^{-x}\cos x$$

$\Rightarrow$ $y_4 + 4y = 0$

$\Rightarrow$ $k_4 = 4$

Differentiating it again four times, we get

$$y_8 + 4y_4 = 0$$

$\Rightarrow$ $y_8 - 16y = 0 \Rightarrow k_8 = -16$

$$y_{12} + 4y_8 = 0 \Rightarrow y_{12} + 64y = 0 \Rightarrow k_{12} = 64$$

Similarly $k_{16} = -256$

12. (a, b)

$$e^{xy} + y\cos x = 2 \quad (1)$$

Putting $x = 0$, we get

$$e^0 + y\cos x = 2$$

$\therefore$ $y = 1$

Differentiating (1) w.r.t. x, we get

$$e^{xy}\left(x\frac{dy}{dx}+y\right) - y\sin x + \cos x\,\frac{dy}{dx} = 0$$

$$\therefore \quad (xe^{xy}+\cos x)\frac{dy}{dx} = y(\sin x - e^{xy}) \quad (2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{y\sin x - ye^{xy}}{xe^{xy}+\cos x}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1\sin 0 - 1e^0}{0+\cos 0}$$

$$= -1$$

Differentiating (2), w.r.t. x, we get

$$(xe^{xy}+\cos x)\frac{d^2y}{dx^2} + \left[e^{xy} + xe^{xy}\left(x\frac{dy}{dx}+y\right) - \sin x\right]\frac{dy}{dx}$$

$$= \frac{dy}{dx}(\sin x - e^{xy}) + y\left[\cos x - e^{xy}\left(y + x\frac{dy}{dx}\right)\right]$$

Putting $x = 0$ and $y = 1$

$$\therefore \quad (0+1)\frac{d^2y}{dx^2} + (e^0)(-1)$$

$$= (-1)(0-1) + 1\cdot(1-1(1))$$

$$\therefore \quad \frac{d^2y}{dx^2} = 2$$

13. (a, d)

$$g(x) = e^{ax} + f(x)$$

$\Rightarrow$ $g'(x) = ae^{ax} + f'(x)$

$\therefore$ $g'(0) = a + f'(0) = a - 5$

Now, $g''(x) = a^2e^{ax} + f''(x)$

$\Rightarrow$ $g''(0) = a^2 + f'(0) = 3 + a^2$

We have $g'(0) + g''(0) = 0$

$\Rightarrow$ $a - 5 + a^2 + 3 = 0$

$\Rightarrow$ $a^2 + a - 2 = 0$

$\Rightarrow$ $(a+2)(a-1) = 0$

$\Rightarrow$ $a = 1, -2$

DPP 4.4

Single Correct Answer Type

1. (d) $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to 0}\frac{f(x)}{x}$$

$$= \lim_{x\to 0}\frac{g(x)}{x^2}$$

$$= \lim_{x\to 0}\frac{g'(x)}{2x}$$

$$= \frac{g''(0)}{2} = \frac{17}{2}$$

2. (b) $f(x+y) = f(x) + f(y) + f(x)f(y),\ \forall x, y \in R$

$\therefore$ $1 + f(x+y) = (1+f(x))(1+f(y))$

Put $g(x) = 1 + f(x)$, we get

$\therefore$ $g(x+y) = g(x)\,g(y)$

$\therefore$ $g(x) = e^{kx}$

$\therefore$ $f(x) = e^{kx} - 1$

$f'(0) = 1$ gives $k = 1$

$\therefore$ $f(x) = e^x - 1$

$\therefore$ $[f(2)] = [e^2 - 1] = 6$

3. (b) Put $x = x$, $y = h$

$$f(x+h) - f(x) = \lambda xh + 3x^2h^2$$

$\therefore$ $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lambda x$

$\therefore$ $f'(x) = \lambda x$

Putting $x = 3$, $y = 2$

$\therefore$ $f(5) = f(3) + \lambda(6) + 3(3)^2(2)^2$

$\therefore$ $48 = 6\lambda + 108$

$\Rightarrow$ $\lambda = -10$

$f'(x) = -10x$

4. (d) $f(xy) = f(x)f(y) - f(x) - f(y) + 2$ (1)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left\{x\left(1+\frac{h}{x}\right)\right\} - f(x)}{h} \quad (x \neq 0 \text{ given})$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-2}{\frac{h}{x}}\cdot\frac{f(x)-1}{x}$$

Putting $x = 1$, $y = 2$ in (1), we get $f(1) = 2$

$\therefore$ $f'(x) = f'(1)\cdot\dfrac{f(x)-1}{x}$

5. (a) We have $\dfrac{f(x+y)-f(x)}{2} = \dfrac{f(y)-a}{2} + xy$

$\therefore$ $f(x+y) = f(x) + f(y) - a + 2xy$

Put $x = y = 0$

$\Rightarrow$ $f(0) = a$

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$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{f(x)+f(h)-a+2xh-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{f(h)-a}{h}+2x$$
$$=\lim_{h\to 0}\frac{f(h)-f(0)}{h}+2x$$
$$=f'(0)+2x$$

$\Rightarrow \quad f'(x)=2x+\sqrt{5a-1-a^2}$

$\Rightarrow \quad f(x)=x^2+(\sqrt{5a-1-a^2})x+c$

Putting $x=0$, we get $c=a$

$\Rightarrow \quad f(x)=x^2+(\sqrt{5a-1-a^2})x+a$
$$=\left(x+\frac{\sqrt{5a-1-a^2}}{2}\right)^2+a-\frac{5a-1-a^2}{4}$$
$$=\left(x+\frac{\sqrt{5a-1-a^2}}{2}\right)^2+\frac{4a-5a+1+a^2}{4}$$
$$=\left(x+\frac{\sqrt{5a-1-a^2}}{2}\right)^2+\frac{a^2-a+1}{4}>0$$

(as $a^2-a+1>0$)

$\therefore \quad f(x)>0\ \forall\ x\in R$

6. (d) $f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{f(3-h)-f(3)}{-h}=5$

$\therefore \quad f(3+h)-f(3)=5h+h=\varepsilon_1$
where $\varepsilon_1\to 0$ when $h\to 0$
and $\quad f(3-h)-f(3)=-5h+h=\varepsilon_2$
where $\varepsilon_2\to 0$ when $h\to 0$
$\therefore \quad f(3+h)=f(3)+5h+h=\varepsilon_1=4+5h+h\varepsilon_1$
$\therefore \quad [f(3+h)]=[4+5h+h\varepsilon_1]=4$ when $h>0$ and h is very small.

$\therefore \quad \lim_{h\to 0}[f(3+h)]=4$

Similarly, $[f(3-h)]=[4-5h+h\varepsilon_2]=[4-h(5-\varepsilon_2)]=3$
When $h>0$ and h is very small implying ε_2 is very small.

$\therefore \quad \lim_{h\to 0}[f(3-h)]=3\neq 4=\lim_{h\to 0}[f(3+h)]$

$\therefore \quad \lim_{x\to 3}[f(x)]$ does not exist.

7. (c) Replace x by $-x$
$\therefore \quad f(2x^2-1)=-2x^3 f(-x)$
$\quad 2x^3 f(x)=-2x^3 f(-x)$
$\Rightarrow \quad f(-x)=-f(x)$
$\therefore \quad f(x)$ or odd function
$\therefore \quad f^{(n)}(0)=0$, when n is even

8. (c) $f(x)=\begin{vmatrix}(x-a)^4 & (x-a)^3 & 1\\ (x-b)^4 & (x-b)^3 & 1\\ (x-c)^4 & (x-c)^3 & 1\end{vmatrix}$

$$f'(x)=\begin{vmatrix}4(x-a)^3 & (x-a)^3 & 1\\ 4(x-b)^3 & (x-a)^3 & 1\\ 4(x-c)^3 & (x-a)^3 & 1\end{vmatrix}+\begin{vmatrix}(x-a)^4 & 3(x-a)^2 & 1\\ (x-b)^4 & 3(x-b)^2 & 1\\ (x-c)^4 & 3(x-c)^2 & 1\end{vmatrix}$$
$$+\begin{vmatrix}(x-a)^4 & (x-a)^3 & 0\\ (x-b)^4 & (x-b)^3 & 0\\ (x-c)^4 & (x-c)^3 & 0\end{vmatrix}$$
$$=3\begin{vmatrix}(x-a)^4 & (x-a)^2 & 1\\ (x-b)^4 & (x-b)^2 & 1\\ (x-c)^4 & (x-c)^2 & 1\end{vmatrix}$$

$\therefore \quad \lambda=3$

9. (b) $\begin{vmatrix}f'(x) & f(x)\\ f''(x) & f'(x)\end{vmatrix}=0$

$\Rightarrow \quad f'(x)f'(x)-f(x)\cdot f''(x)=0$

$\Rightarrow \quad \frac{(f'(x))^2-f(x)f''(x)}{(f'(x))^2}=0$

$\Rightarrow \quad \frac{d}{dx}\left[\frac{f(x)}{f'(x)}\right]=0$

$\Rightarrow \quad \frac{f(x)}{f'(x)}=C$ (i)

Put $x=0$, $\frac{f(0)}{f'(0)}=C\Rightarrow C=\frac{1}{2}$

Hence $\frac{f(x)}{f'(x)}=\frac{1}{2}$

From (i), $2f(x)=f'(x)$

$\therefore \quad \frac{f'(x)}{f(x)}=2$

$\therefore \quad \frac{d}{dx}(\log f(x))=2$

$\therefore \quad \log(f(x))=2x+k$
Putting $x=0$, we get $k=0$
$\Rightarrow \quad f(x)=e^{2x}$

$\Rightarrow \quad \lim_{x\to 0}\frac{e^{2x}-1}{x}=2$

10. (d) Let degree of $f(x)$ is n;
Degree of $f'(x)$ is $n-1$
Degree of $f''(x)$ is $(n-2)$
Hence $n=(n-1)+(n-2)=2n-3$
$\therefore \quad n=3$
$\therefore \quad f(x)=ax^3+bx^2+cx+d,\quad (a\neq 0)$
$\quad f'(x)=3ax^2+2bx+c$
$\quad f''(x)=6ax+2b$
$\Rightarrow \quad ax^3+bx^2+cx+d=(3ax^2+2bx+c)(6ax+2b)$
$\Rightarrow \quad 18a^2=a$
$\Rightarrow \quad a=1/18$

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11. (a) We have $f'(x) > 0$ and $f''(x) > 0$

Let $g(x) = f^{-1}(x)$

$f(g(x)) = x$

$f'(g(x)) \cdot g'(x) = 1$

$$g'(x) = \frac{1}{f'(g(x))}$$

$\therefore \quad g'(x) > 0$

$$g''(x) = -\frac{1}{(f'(g(x)))^2} f''(g(x)) g'(x)$$

$< 0 \qquad (\because f''(x) < 0)$

$$\therefore \quad \frac{d^2(f^{-1}(x))}{dx^2} = g''(x) < 0$$

12. (b) $x_1 = 2R\cos A, x_2 = 2R\cos B, x_3 = 2R\cos C$.

$$\frac{dx_1}{dA} = -2R\sin A$$

Also $a = 2R\sin A \Rightarrow \frac{da}{dA} = 2R\cos A$

So, $\quad \frac{dx_1}{da} = -\tan A, \frac{dx_2}{db} = -\tan B, \frac{dx_3}{dc} = -\tan C$

Now $\quad \tan A + \tan B + \tan C \geq 3\sqrt{3}$

So, $\quad \left(\frac{dx_1}{da} + \frac{dx_2}{db} + \frac{dx_3}{dc}\right) \leq -3\sqrt{3}$

13. (a) $(fg)' = f'g'$

$\therefore \quad fg' + f'g = f'g'$

$\therefore \quad xg' + 3g = 3g'$

$\therefore \quad (x-3)g' + 3g = 0$

$\therefore \quad (x-3)^3 g' + 3(x-3)^2 g = 0$

$\therefore \quad (g(x-3)^3)' = 0$

$\therefore \quad (x-3)^3 g(x) = c$

$$\therefore \quad g(x) = \frac{1}{(x-3)^3} \quad (g(4) = 1)$$

14.(c) g is inverse of f

$\therefore \quad fog(x) = x$

$\therefore \quad f'(g(x)) \cdot g'(x) = 1$

$$\therefore \quad g'(x) = \frac{1}{f'(g(x))} \qquad (1)$$

$$= \sqrt{1 + g^3(x)}$$

$$\therefore \quad g''(x) = \frac{3g^2(x) \cdot g'(x)}{2\sqrt{1 + g^3(x)}}$$

$$= \frac{3g^2(x)}{2} \qquad \text{(using (1))}$$

15. (c) $\qquad (i)$

$3f(x+y) = f(x) f(y)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(x) \cdot f(h)}{3} - f(x)}{h} \qquad \text{(using } f \text{ rule)}$$

$$= \lim_{h \to 0} \frac{f(x) \cdot f(h) - 3f(x)}{3h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f(x)}{3} \cdot \frac{[f(h) - 3]}{h}$$

Put $\quad x = 0, y = 1$

$\therefore \quad 3f(1) = f(0)f(1)$

$\therefore \quad f(0) = 3$

$$\therefore \quad f'(x) = \frac{f(x)}{3} \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$\Rightarrow \quad f'(x) = \frac{f(x)}{3} f'(0);$$

$$\Rightarrow \quad 3 \cdot \frac{f'(x)}{f(x)} = f'(0) = k \text{ (say)}$$

$\Rightarrow$ Integrating, we get $3\log(f(x)) = kx + C$;

Using $f(0) = 3 \Rightarrow 3\log 3 = C$

$\Rightarrow \quad 3\log(f(x)) = kx + 3\log 3$

$$\Rightarrow \quad 3\log\left(\frac{f(x)}{3}\right) = kx;$$

Using $f(1) = 6 \quad \Rightarrow \quad 3\log 2 = k$

$\Rightarrow \quad f(x) = 3 \cdot 2^x$

$\Rightarrow \quad f(2) = 12$

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CHAPTER 5

DPP 5.1

Single Correct Answer Type

1. (c) $t^2+3t-8=2 \Rightarrow t=2,-5$
$2t^2-2t-5=-1 \Rightarrow t=2,-1$

$\Rightarrow \quad t=2, \frac{dy}{dx}=\frac{4t-2}{2t+3} \Rightarrow \left(\frac{dy}{dx}\right)_{t=2}=\frac{6}{7}$

$\therefore$ Equation of normal $y+1=\frac{-7}{6}(x-2)$

2. (d) $f(x)=x^3+ax$ and $g(x)=bx^2+c$ pass through the point $(-1, 0)$

$\Rightarrow \quad f(-1)=0;\ g(-1)=0;$

$\Rightarrow \quad -1-a=0$ and $b+c=0$ (i)

Also curves have common tangent at this point

$\Rightarrow \quad f'(-1)=g'(-1)$

$\Rightarrow \quad a+3=-2b$ (ii)

From (i) and (ii)

$a=-1;\ b=-1;\ c=1;$

$\Rightarrow \quad$ hence $a+b+c=-1$

3. (b) According to the question we must have

$f'(x)=0$ and $f''(x)=0$ for the same $x=x_0$

now $f'(x)=4x^3+2bx+8$

$\therefore \quad f'(x_0)=2[2x_0^3+bx_0+4]=0$ (i)

and $f''(x_0)=2[6x_0^2+b]=0$ (ii)

From (ii) $b=-6x_0^2$

Substituting this value of b in (i)

$2x_0^3+(-6x_0^2)+4=0 \Rightarrow 4x_0^3=4$

$\Rightarrow \quad x_0=1.$

$\Rightarrow \quad b=-6$

4. (a) Let $(3, \alpha)$ be the point on $y=g(x) \Rightarrow (\alpha, 3)$ lies on $y=f(x)$
$\Rightarrow \alpha=1$

Also $x=f(g(x))$

$\Rightarrow \quad g'(3)=\frac{1}{f'(g(3))}=\frac{1}{f'(1)}=\frac{1}{4}$

$\therefore \quad$ Tangent is $y-1=\frac{1}{4}(x-3)$ or $x-4y+1=0.$

5. (b) $x=t+e^{at};\ y=-t+e^{at}$

$\frac{dx}{dt}=1+ae^{at};\quad \frac{dy}{dt}=-1+ae^{at},\ \frac{dy}{dx}=\frac{-1+ae^{at}}{1+ae^{at}}$

at the point A, $y=0$ and $\frac{dy}{dx}=0$ for some $t=t_1$

$\therefore \quad ae^{at_1}=1$ (i);

also $0=-t_1+e^{at_1}; \quad \therefore \quad e^{at_1}=t_1$ (ii),

putting this value in (i)

we get, $at_1=1 \quad \Rightarrow \quad t_1=\frac{1}{a};$

now from (i) $ae=1 \quad \Rightarrow \quad a=\frac{1}{e}$

hence $x_A=t_1+e^{at_1}=e+e=2e$

$\Rightarrow$ Point A is $(2e, 0)$

6. (a) $x=2t^3, y=3t^2$

$\therefore \quad$ tangent at t is $x-yt=-t^3$

and normal at t_1 is, $xt_1+y=2t_1^4+3t_1^2$

Since both equations represent same straight line,

$$\frac{1}{t_1}=-t=\frac{-t^3}{2t_1^4+3t_1^2}$$

$\Rightarrow \quad t^6-3t^2-2=0 \Rightarrow t^2=2 \Rightarrow t=\pm\sqrt{2}$

$\therefore \quad$ lines are $x=\pm\sqrt{2}\,(y-2)$

7. (c) $2y\frac{dy}{dx}=(2-x)^2-2x(2-x)$

So $\left.\frac{dy}{dx}\right|_{(1,1)}=-\frac{1}{2}$

Therefore, the equation of tangent at $(1, 1)$ is

$$y-1=-\frac{1}{2}(x-1)$$

$\Rightarrow \quad y=\frac{-x+3}{2}$

The intersection of the tangent and the curve is given by

$(1/4)(-x+3)^2=x(4+x^2-4x)$

$\Rightarrow \quad 4x^3-17x^2+22x-9=0$

$\Rightarrow \quad (x-1)(4x^2-13x+9)=0$

$\Rightarrow \quad (x-1)^2(4x-9)=0$

Since $x=1$ is already the point of tangency, $x=9/4$ and

$y^2=\frac{9}{4}\left(2-\frac{9}{4}\right)^2=\frac{9}{64}.$

Thus, the required point is $(9/4, 3/8)$.

8. (c) $y=ax^4+bx^3+cx+d$

$\Rightarrow \quad dy/dx=4ax^3+3bx^2+c$

Point $(0, 1)$ satisfies the curve $\Rightarrow d=1$

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Also $\left(\frac{dy}{dx}\right)_{(0,1)} = 0 \Rightarrow c = 0$ (1)

$(-1, 0)$ satisfies the curve $\Rightarrow a - b - c + 1 = 0$ (2)

Also $\left(\frac{dy}{dx}\right)_{(-1,0)} = 0 \Rightarrow c = 0$

$\Rightarrow$ $-4a + 3b = 0$ and $a - b = -1$

$\Rightarrow$ $a = 3, b = 4$

$\Rightarrow$ The curve is $y = 3x^4 + 4x^3 + 1$

$\Rightarrow$ $dy/dx = 12x^3 + 12x^2$

For $dy/dx < 0$ we have $x^3 + x^2 < 0 \Rightarrow x^2 (x + 1) < 0 \Rightarrow x < -1$

9. (d) $x = 2t - t^2$ (i)

and $y = t + t^2$ (ii)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+2t}{2-2t}$$

Slope of tangent using the points $(1, 1)$ and $(2t - t^2, t + t^2)$

$$\frac{dy}{dx} = \frac{t+t^2-1}{2t-t^2-1} \quad \text{(iii)}$$

Equating (ii) and (iii), $\frac{t+t^2-1}{2t-t^2-1} = \frac{1+2t}{2-2t}$

$\Rightarrow$ $3t^2 - 4t + 1 = 0$

$\Rightarrow$ $t = 1, \frac{1}{3}$

For $t = 1$, point is $P(1, 2)$ and for $t = \frac{1}{3}$, point is $Q\left(\frac{5}{9}, \frac{4}{9}\right)$.

$\therefore$ $PQ = \frac{2\sqrt{53}}{9}$

10. (b) Slope of line $(4 - t)x + ty + (a^3 - 1) = 0$ is $\frac{t-4}{t}$.

For $xy = 1$, $\frac{dy}{dx} = \frac{-y}{x} = \frac{-1}{x^2}$

$\therefore$ Slope of normal $= x^2 = \frac{t-4}{t}$

As $x^2 > 0$, $\frac{t-4}{t} > 0$

$\therefore$ $t \in (-\infty, 0) \cup (4, \infty)$

11. (b) $\frac{dy}{dx} = \frac{4t}{3}$

Tangent is $y - at^4 = \frac{4t}{3}(x - at^3)$

x-intercept $= \frac{at^3}{4}$

y-intercept $= -\frac{at^4}{3}$

the point of intersection of tangent with the axes are $\left(\frac{at^3}{4}, 0\right)$ and $\left(0, -\frac{at^4}{3}\right)$

$\therefore$ $\frac{m}{n} = -\frac{3}{4}$

Multiple Correct Answers Type

12. (b, c)

Let the tangent line be $y = ax + b$

The equation for its intersection with the upper parabola is

$x^2 + 1 = ax + b \Rightarrow x^2 - ax + (1 - b) = 0$

This has equal roots when $a^2 - 4(1 - b) = 0$ or $a^2 + 4b = 4$ (1)

For the lower parabola

$ax + b = -x^2 \Rightarrow x^2 + ax + b = 0$

This has equal roots when $a^2 - 4b = 0$ (2)

From (1) and (2) $8b = 4$ or $b = 1/4$

Add $2a^2 = 4$ or $a = \pm\sqrt{2}$

The tangent lines are $y = \sqrt{2}x + \frac{1}{2}$ and $y = -\sqrt{2}x + \frac{1}{2}$.

13. (a, b)

It can be easily checked that function is continuous and differentiable at $x = 0$.

Now $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ (i)

$$g'(0) = \lim_{t\to 0}\left(\frac{g(t)-g(0)}{t-0}\right) = \lim_{t\to 0}\left(\frac{\frac{1}{t}\sin t^2 - 0}{t}\right) = 1$$

$$f'(0) = \lim_{t\to 0}\left(\frac{f(t)-f(0)}{t-0}\right) = \lim_{t\to 0}\frac{2t + t^2\sin\frac{1}{t}}{t} = 2 + 0 = 2$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{1}{2}$

So the equation of tangent $y - 0 = \frac{1}{2}(x - 0)$

and equation of normal $y - 0 = -2(x - 0)$

14. (a, c)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\cos^2 t \sin t}{2a\cos t + a\cos^3 t - 2a\sin^2 t\cos t}$$

$$= \frac{3a\cos^2 t\sin t}{2a\cos t(1-\sin^2 t) + a\cos^3 t}$$

$$= \frac{3a\cos^2 t\sin t}{2a\cos^3 t + a\cos^3 t}$$

$$= \frac{3a\cos^2 t\sin t}{3a\cos^3 t}$$

$\frac{dy}{dx} = \tan t$, equation of normal is

$$y + a\cos^3 t = -\frac{\cos t}{\sin t}(x - 2a\sin t - a\sin t\cos^2 t)$$

$\therefore$ $y\sin t + a\cos^3 t\sin t = -x\cos t + 2a\sin t\cos t + a\sin t\cos^3 t$

$$x\cos t + y\sin t = 2a\sin t\cos t$$

It meets axis at $A(2a\sin t, 0)$ and $B(0, 2a\cos t)$

Then $AB = 2a$.

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15. (a, b, d)

$\frac{dy}{dx} = 3ax^2 + 2bx + c = \tan 45° = 1$ at $x = 0$

$\therefore \quad c = 1$

$\frac{dy}{dx}$ at $(1, 0) = 3a + 2b + c = 0$ as x-axis is tangent.

$\therefore \quad 3a + 2b + 1 = 0 \quad \because c = 1$

$(1, 0)$ lies on curve,

$\therefore \quad a + b + 1 = 0$

Solving we get $a = 1, b = -2$

$\therefore \quad f(x) = x^3 - 2x^2 + x$

and $f'(x) = 3x^2 - 4x + 1$

DPP 5.2

Single Correct Answer Type

1. (c) $x^2 - 4y^2 = 4$

$\therefore \quad 2x - 8y\,y' = 0$ (i)

Slope of normal is 1.

$\therefore$ Slope of tangent is -1.

$\therefore \quad x + 4y = 0$ (ii)

Also solving (i) and (ii), $16y^2 - 4y^2 = 4$

$\therefore \quad y^2 = 1/3$

$\therefore$ Point on the curve is $\left(\mp\frac{4}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

Length of normal $= \left|y\frac{dx}{dy}\right|$

Given $\frac{1}{\sqrt{k}} = \frac{1}{\sqrt{3}}$

$\therefore \quad k = 3$

2. (b) Point of intersection of curves is $(1, 0)$.

$y = 3^{x-1} \ln x$

$\therefore \quad y' = \frac{3^{x-1}}{x} + 3^{x-1}\log_e 3.\log_e x$

$\therefore \quad m_1 = 1$

$y = x^x - 1$

$\therefore \quad \frac{dy}{dx} = x^x(1 + \log_e x)$

$\therefore \quad m_2 = 1$

$\therefore$ Angle between curves is given by

$\tan\theta = \left|\frac{1-1}{1+1}\right| = 0 \quad \Rightarrow \quad \theta = 0 \quad \Rightarrow \quad \cos 0° = 1$

3. (c) $x^2 + y^2 = a^2\sqrt{2}$ and $x^2 - y^2 = a^2$

$\therefore \quad \frac{dy}{dx} = \frac{-x}{y}; \frac{dy}{dx} = \frac{x}{y}$

Angle between curves is given by

$$\theta = \left|\tan^{-1}\frac{\frac{x}{y} + \frac{x}{y}}{1 - \frac{x^2}{y^2}}\right| = \left|\tan^{-1}\frac{2xy}{y^2 - x^2}\right|$$

Squaring and subtracting the equations of given curves,

$4x^2y^2 = a^4$

$\therefore \quad 2xy = \pm a^2$

$\therefore \quad \theta = \left|\tan^{-1}\frac{a^2}{a^2}\right|$

$= \tan^{-1}(1)$

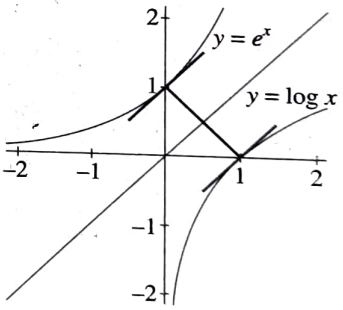
$= \frac{\pi}{4}$

4. (b) Since $y = e^x$ and $y = \log_e x$ are inverse to each other their graphs are symmetrical about $y = x$.

Minimum distance between the curves is the distance between the points on the curves where tangent is parallel to $y = x$ or slope of tangent is 1.

Now for $y = e^x$, $y' = e^x$, for $e^x = 1$, $x = 0$, so the point on the curve $y = e^x$ is $(0, 1)$ and symmetric point on the curve $y = \log_e x$ is $(1, 0)$.

Distance between these points is $\sqrt{2}$.



5. (d) $y_1 = \sqrt{2}\sin\left(x_1 + \frac{\pi}{4}\right), \frac{dy}{dx} = \frac{y_1}{x_1}$ where (x_1, y_1) is point on the curve

$\therefore \quad \frac{y_1^2}{x_1^2} = 2\cos^2\left(x_1 + \frac{\pi}{4}\right) = 2\left(\frac{-y_1^2}{2} + 1\right)$

$\Rightarrow$ Locus of (x_1, y_1) is $\frac{2}{y^2} - \frac{1}{x^2} = 1$

6. (a) Slope of $3x + 2y = 1$ is $\frac{-3}{2}$

$\Rightarrow \quad f'\left(\frac{1}{2}\right) = \frac{-3}{2}$

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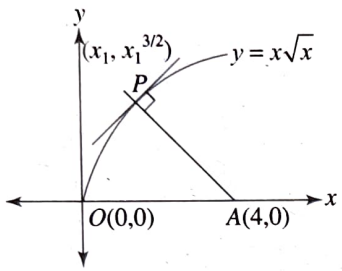
$$\therefore \quad \lim_{x\to 0}\frac{x(x-1)}{f\left(\frac{e^{2x}}{2}\right)-f\left(\frac{e^{-2x}}{2}\right)}$$

$$=\lim\frac{2x-1}{e^{2x}\cdot f'\left(\frac{e^{2x}}{2}\right)+e^{-2x}f'\left(\frac{e^{-2x}}{2}\right)}$$

$$=\frac{-1}{f'\left(\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)}=\frac{1}{3}$$

7. (a)



AP will be least if AP is normal at point P.

$\therefore$ Slope of $AP \times$ Slope of tangent at point $P = -1$

Slope of tangent at point $P = \left(\frac{dy}{dx}\right)_P = \frac{3}{2}\sqrt{x_1}$

$$\therefore \quad \frac{3}{2}\sqrt{x_1}\cdot\frac{x_1^{3/2}-0}{x_1-4}=-1$$

$$\Rightarrow \quad 3x_1^2=8-2x_1$$

$$\Rightarrow \quad 3x_1^2+2x_1-8=0$$

$$\Rightarrow \quad (x_1+2)(3x_1-4)=0$$

$$\Rightarrow \quad x_1=\frac{4}{3}$$

$$\Rightarrow \quad \text{Point } P\left(\frac{4}{3},\left(\frac{4}{3}\right)^{3/2}\right)$$

$$\Rightarrow \quad OP=\sqrt{\frac{112}{27}}$$

8. (d) $y = 3x + 3$ is normal to the curve at $x = 0$

Slope of normal is 3.

$\therefore$ Slope of tangent at $x = 0$ is $f'(0) = -\frac{1}{3}$

Now $\lim_{x\to 0}\frac{x^2}{\{f(x^2)-5f(4x^2)+4f(7x^2)\}}\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 0}\frac{2x}{2xf'(x^2)-40xf'(4x^2)+56xf'(7x^2)}$$

$$=\lim_{x\to 0}\frac{1}{f'(0)-20f'(0)+28f'(0)}$$

$$=\frac{1}{9f'(0)}=\frac{1}{9\left(-\frac{1}{3}\right)}=-\frac{1}{3}$$

Multiple Correct Answers Type

9. (a, b)

$$L_{ST}=\left|\frac{y}{m}\right|,\ L_{SN}=|ym|$$

$$L_T=\left|\frac{y\sqrt{1+m^2}}{m}\right|,\ L_N=|y\sqrt{1+m^2}|$$

where $m = \frac{dy}{dx}$ at point $P = (x, y)$ on the curve $y = f(x)$

Now $\frac{L_{ST}}{L_{SN}}=\frac{1}{m^2}=\left(\frac{L_T}{L_N}\right)^2$ and $L_{ST}L_{SN}=y^2$

10. (a, b, c, d)

(a) $16x^2 + y^2 = c \Rightarrow \frac{dy}{dx} = m_1 = -\frac{16x}{y}$

$$y^{16}=kx \Rightarrow \frac{dy}{dx}=m_2=\frac{k}{16y^{15}}$$

$$m_1m_2=-\frac{16x}{y}\cdot\frac{k}{16y^{15}}$$

$$=-\frac{x}{y^{16}}\cdot k$$

$$=-\frac{x}{y^{16}}\cdot\frac{y^{16}}{x}=-1$$

$\Rightarrow$ Curves are orthogonal.

(b) $y = x + ce^{-x}$

$$\Rightarrow \quad \frac{dy}{dx}=1-ce^{-x}=1-(y-x)=-(y-x-1)$$

$$x+2=y+ke^{-y}$$

$$\Rightarrow \quad \frac{dy}{dx}-k\frac{dy}{dx}e^{-y}=1$$

$$\Rightarrow \quad \frac{dy}{dx}[1-ke^{-y}]=1$$

or $\quad [1-(x+2-y)]\frac{dy}{dx}=1$

$$\Rightarrow \quad \frac{dy}{dx}=m_2=\frac{1}{y-x-1}$$

$$\Rightarrow \quad m_1m_2=-1$$

$\Rightarrow$ Curves are orthogonal.

(c) $y = cx^2$

$$\Rightarrow \quad \frac{dy}{dx}=2cx=2x\cdot\frac{y}{x^2}=\frac{2y}{x}=m_1$$

and $\quad x^2+2y^2=k$

$$\Rightarrow \quad 2x+4y\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{x}{2y}=m_2$$

$$\Rightarrow \quad m_1m_2=-1$$

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$\Rightarrow$ Curves are orthogonal.

(d) $x^2 - y^2 = c \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y} = m_1$

$xy = k \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x} = m_2$

$\therefore \quad m_1 m_2 = -1$

$\Rightarrow$ Curves are orthogonal.

DPP 5.3

Single Correct Answer Type

1. (c) $u = \sqrt{x^2+16}$

$\therefore \quad \dfrac{du}{dx} = \dfrac{2x}{2\sqrt{x^2+16}} = \dfrac{x}{\sqrt{x^2+16}}$

$v = \dfrac{x}{x-1}$

$\Rightarrow \quad \dfrac{dv}{dx} = \dfrac{-1}{(x-1)^2}$

$\therefore \quad \dfrac{du}{dv} = \dfrac{du/dx}{dv/dx} = \dfrac{-12}{5}$

2. (c) Eccentricity of ellipse $= \dfrac{1}{2}$

$\dfrac{de}{dt} = -0.1$ (given)

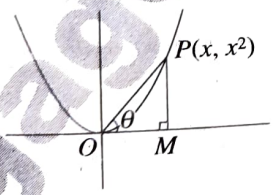
Eccentricity of auxiliary circle = 0

$\Rightarrow \quad \int_{1/2}^{0} de = -0.1 \int_{0}^{T} dt$

T is time at which it will touch the auxiliary circle.

$\Rightarrow \quad -\dfrac{1}{2} = -0.1\,(T-0) \quad \therefore T = 5$ seconds.

3. (a) $y = x^2$



$\tan\theta = \dfrac{x^2}{x} = x$

$\Rightarrow \quad \sec^2\theta \dfrac{d\theta}{dt} = \dfrac{dx}{dt}$

$\Rightarrow \quad \dfrac{d\theta}{dt} = 10 \times \cos^2\theta = 10 \times \dfrac{1}{10} = 1$ {at $x = 3$ m}

4. (b) $V = \dfrac{4}{3}\pi r^3$

$\therefore \quad \dfrac{dV}{dt} = \dfrac{4}{3}\pi \cdot 3r^2 \dfrac{dr}{dt}$

Given $\dfrac{dV}{dt} = \dfrac{dr}{dt}$

$\therefore \quad r^2 = \dfrac{1}{4\pi} \Rightarrow r = \dfrac{1}{2\sqrt{\pi}}$

$\therefore \quad k = 2$

5. (d) Given that $\dfrac{dV}{dt} = 2$

$\Rightarrow \quad \dfrac{d}{dt}\left(\dfrac{1}{3}\pi r^3\right) = 2$

$\Rightarrow \quad \pi r^2 \cdot \dfrac{dr}{dt} = 2$

$\Rightarrow \quad \dfrac{dr}{dt} = \dfrac{2}{\pi r^2}$

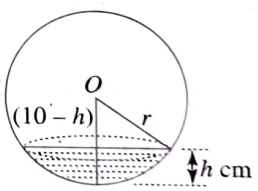
$\Rightarrow \quad \dfrac{d}{dt}(2\pi r) = \dfrac{4}{r^2}$ (i)

When $h = 2$ m

$\Rightarrow \quad r = 2$ m

Hence, $\dfrac{d(2\pi r)}{dt} = \dfrac{4}{4} = 1$ m/s

6. (c)



$\dfrac{dh}{dt} = -2$, $r = 10$ cm

We have to find $\dfrac{dx}{dt}$ when $h = 4$, where x is the radius of the top surface.

From the figure $r^2 = x^2 + (10-h)^2$

$\therefore \quad 2x\dfrac{dx}{dt} = 2(10-h)\dfrac{dh}{dt}$

$\Rightarrow \quad \dfrac{dx}{dt} = \dfrac{(10-h)}{x}(-2)$

$\Rightarrow \quad \dfrac{dx}{dt} = \dfrac{-2(10-4)}{x} = -\dfrac{12}{x}$ (i)

When $h = 4$, then $x^2 = 10^2 - 6^2 = 64$ or $x = 8$.

$\therefore \quad \dfrac{dx}{dt} = \dfrac{-12}{8} = -\dfrac{3}{2}$

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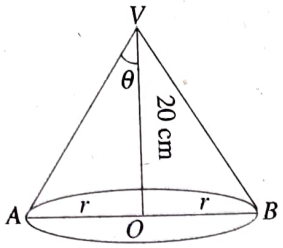
7. (b)



Let θ be the semi-vertical angle and r be the radius of the cone at time t.

Then, $r = 20 \tan\theta$

$$\Rightarrow \quad \frac{dr}{dt} = 20\sec^2\theta\frac{d\theta}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = 20\sec^2 30° \times 2 \left[\because \theta = 30° \text{ and } \frac{d\theta}{dt} = 2\right]$$

$$\Rightarrow \quad \frac{dr}{dt} = 20 \times \frac{4}{3} \times 2 \text{ cm/s} = \frac{160}{3} \text{ cm/s}$$

8. (d) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{a\sin\theta}{a(1+\cos\theta)} = \tan\dfrac{\theta}{2}$

$\therefore$ The rate of change of the slope, i.e.,

$$\frac{dy}{dt} = \frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{d\theta}{dt} = \frac{k}{2}\sec^2\frac{\theta}{2}$$

$$\therefore \text{ Required rate} = \frac{k}{2}\cdot\sec^2\frac{\pi}{6} = \frac{k}{2}\cdot\left(\frac{2}{\sqrt{3}}\right)^2$$

Comprehension Type

9. (a) Let $WB = x$ at time t. Then

$x/13 = \cos\theta$

$$\therefore \quad \frac{dx}{dt} = 13(-\sin\theta)\frac{d\theta}{dt}$$

When $WB = 12$, then

$$5 = -13\cdot\frac{5}{13}\cdot\frac{d\theta}{dt}$$

$$\therefore \quad \frac{d\theta}{dt} = -1 \text{ rad/s}$$

10. (b) $WM = 15$ m

Height of the lamp post at time $t = \sqrt{169 - x^2} + 10$ m

Let ℓ be the length of shadow at time t. Then

$$\frac{\sqrt{169-x^2}+10}{1.5} = \frac{15+\ell}{\ell} = \frac{15}{\ell} + 1$$

$$\therefore \quad \sqrt{169-x^2} + 10 = \frac{22.5}{\ell} + 1.5$$

$$\therefore \quad \frac{1}{2}\frac{-2x}{\sqrt{169-x^2}}\frac{dx}{dt} = -\frac{22.5}{\ell^2}\frac{d\ell}{dt}$$

$$\therefore \quad \frac{-12}{5}\times 5 = \frac{-22.5}{\ell^2}\cdot\frac{d\ell}{dt} \Rightarrow \frac{d\ell}{dt} = \frac{12\ell^2}{22.5}$$

By the given condition, we have

$$\frac{15}{\ell} + 1 = \frac{15}{1.5} = 10$$

$$\frac{d\ell}{dt} = \frac{12}{22.5}\times\left(\frac{15}{9}\right)^2 = \frac{120}{81} = \frac{40}{27}$$

DPP 5.4

Subjective Type

1. Consider $f(x) = (x - \lambda)\log x$ for $1 \le x \le \lambda$.

The function f is continuous on $[1, \lambda]$

and $f'(x) = \log x + \dfrac{x-\lambda}{x}$ for all $x \in (1, \lambda)$

Also, $f(1) = 0$ and $f(\lambda) = 0$.

So, by Rolle's theorem, there is $c \in (1, \lambda)$ such that $f'(c) = 0$.

$$\Rightarrow \quad \log c + \frac{c-\lambda}{c} = 0$$

$$\Rightarrow \quad c\log c + (c - \lambda) = 0$$

$$\Rightarrow \quad c\log c + c = \lambda$$

Hence, $\log x + x = \lambda$ has at least one solution in $[1, \lambda]$.

2. $\dfrac{f'(c)}{f(a)-f(c)} + \dfrac{g'(c)}{g(b)-g(c)} = 1$

Put $c = x$

$$\Rightarrow \quad \frac{f'(x)}{f(x)-f(a)} + \frac{g'(x)}{g(x)-g(b)} = -1$$

$$\Rightarrow \quad \frac{d}{dx}(\log_e(f(x)-f(a)) + \frac{d}{dx}(\log_e(g(x)-g(b)) = -1$$

$$\Rightarrow \quad d[(\log_e(f(x)-f(a))\,(g(x)-g(b))] = -dx$$

$$\Rightarrow \quad (\log_e(f(x)-f(a))\,(g(x)-g(b))] = -x + k$$

$$\Rightarrow \quad (f(x)-f(a))\,(g(x)-g(b)) = e^{-x+k}$$

$$\Rightarrow \quad (f(x)-f(a))\,(g(x)-g(b))\,e^x - e^k = 0$$

Now let $h(x) = (f(x)-f(a))(g(x)-g(b))e^x - e^k$

$$h(a) = h(b) = -e^k$$

Then by Rolle's Theorem, there exists atleast one $c \in (a, b)$ such that

$$h'(c) = 0$$

$$\Rightarrow \quad \frac{f'(c)}{f(a)-f(c)} + \frac{g'(c)}{g(b)-g(c)} = 1$$

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Single Correct Answer Type

3. (c) $f(x) = x - 1,\ 1 \le x \le 2$

$g(x) = x - 1 + b \sin \frac{\pi}{2} x,\ 1 \le x \le 2$

$f(1) = 0; f(2) = 1 \Rightarrow$ Rolle's theorem is not applicable to 'f but LMVT is applicable to f

($\because$ $x - 1$ is continuous and differentiable in $[1, 2]$ and $(1, 2)$ respectively)

Now $g(1) = b$; $g(2) = 1$ and

Function $x - 1$, $\sin \frac{\pi}{2}$ are both continuous in $[1, 2]$ and $(1, 2)$

$\therefore$ For Rolle's theorem to be applicable to g, we must have $b = 1$

4. (c) $\frac{f(b) - f(a)}{b - a} = f'\left(\frac{1}{2}\right)$

$\Rightarrow \quad a + b = 1$

5. (b) Let $u \in (a, c)$, $v \in (c, b)$

Then by LMVT on (a, c), (c, b), we have

$f'(u) = \frac{f(c) - f(a)}{c - a}$, $f'(v) = \frac{f(b) - f(c)}{b - c}$

But $u < v$ and $f''(x) < 0,\ \forall\ x \in (a, b)$

i.e. $f'(u) > f'(v)$

$\Rightarrow \quad \frac{f(c) - f(a)}{f(b) - f(c)} > \frac{c - a}{b - c}$

6. (a) Let $f(x) = x^{1/3}$

$\Rightarrow \quad f'(x) = \frac{1}{3x^{2/3}}$

Applying LMVT in $[a, a + 1]$, we get one $c \in (a, a + 1)$

$f'(c) = \frac{f(a+1) - f(a)}{a + 1 - a}$

$\Rightarrow \quad \sqrt[3]{a+1} - \sqrt[3]{a} = \frac{1}{3c^{2/3}} < \frac{1}{3a^{2/3}} \le \frac{1}{3n^2}$

$\Rightarrow \quad \sqrt[3]{a+1} - \sqrt[3]{a} < \frac{1}{3n^2}\ \forall a \ge n^3$

7. (d) Given $f'(1) = 1$;

And $f(2x) = f(x)$

$\Rightarrow \quad 2f'(2x) = f'(x)$

Putting $x = 1$, $f'(2) = \frac{f'(1)}{2} = \frac{1}{2}$

Putting $x = 2$, $f'(4) = \frac{f'(2)}{2} = \frac{1}{4}$

Now applying LMVT for $y = f'(x)$ in $[2, 4]$

We get $f''(c) = \frac{f'(4) - f'(2)}{2} = \frac{\frac{1}{4} - \frac{1}{2}}{2} = -\frac{1}{8}$

Multiple Correct Answers Type

8. (a, b, c)

$$f(x) = \begin{vmatrix} 1 & 1 & 1 \\ 3 - x & 5 - 3x^2 & 3x^3 - 1 \\ 2x^2 - 1 & 3x^5 - 1 & 7x^8 - 1 \end{vmatrix}$$

Clearly $f(0) = f(1) = 0$

Hence, $f(x) = 0$ atleast two real roots.

Also for atleast one $x \in (0, 1)$, $f'(x) = 0$ (Rolle's theorem)

Obviously function is many-one.

9. (a, b)

All four functions $\tan^{-1}x$, $\sin^{-1}x$, $\cos^{-1}x$, $\cot^{-1}x$ are continuous and differentiable in their domain.

So by L. M. V. T.

(a) $\frac{\tan^{-1}x - \tan^{-1}y}{x - y} = \frac{1}{1 + c^2} \le 1$

(b) $\frac{\sin^{-1}x - \sin^{-1}y}{x - y} = \frac{1}{\sqrt{1 - c^2}} \ge 1$

(c) $\frac{\cos^{-1}x - \cos^{-1}y}{y - x} = \frac{1}{\sqrt{1 - c^2}} \ge 1$

(d) $\frac{\cot^{-1}x - \cot^{-1}y}{y - x} = \frac{1}{1 + c^2} \le 1$

Comprehension Type

10. (c)

$\phi(x) = f(b) - f(x) - (b - x) f'(x) - (b - x)^2 A$

$\phi(b) = 0$

$\phi(a) = f(b) - f(a) - (b - a) f'(a) - (b - a)^2 A$

Since Rolle's theorem is applicable

$\therefore \quad A = \frac{f(b) - f(a) - (b - a) f'(a)}{(b - a)^2}$ (i)

$\phi'(x) = -f'(x) + f'(x) - (b - x) f''(x) + 2(b - x) A$

$\therefore$ There exists some number $c \in (a, b)$ such that

$0 = \phi'(c) = -(b - c) f''(c) + 2(b - c) A$

i.e. $A = \frac{1}{2} f''(c)$ (ii)

From (i) and (ii), $f(b) - f(a) - (b - a) f'(a) = \frac{1}{2}(b - a)^2 f''(c)$

i.e. $f(b) = f(a) + (b - a) f'(a) + \frac{1}{2}(b - a)^2 f''(c)$

$\therefore \quad \lambda = \frac{1}{2}$

11. (c) $f(1 + h) = f(1) + hf'(1) + \frac{1}{2} h^2 f''(c)$

$\therefore \quad f(1 + h) = 1 + 3h^2 c$

$\therefore \quad \frac{f(1 + h) - f(1)}{h^2} = 3c$

$\therefore \quad \lambda = 3$

12. (b) $\sin(\alpha + h) = \sin\alpha + h\cos\alpha + \frac{1}{2} h^2(-\sin(\alpha + th))$

$\therefore \quad \frac{\sin(\alpha + h) - \sin\alpha - h\cos\alpha}{h^2} = -\frac{1}{2}\sin(\alpha + th)$

$\therefore \quad \alpha = -\frac{1}{2}$

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CHAPTER 6

DPP 6.1

Single Correct Answer Type

1. (b) $f(x) = x \sin x + \cos x + \cos^2 x$

$\therefore \quad f'(x) = \sin x + x \cos x - \sin x - 2 \cos x \sin x$

$= \cos x(x - 2 \sin x)$

For $x \in (0, \pi/2)$, $x < 2 \sin x$.

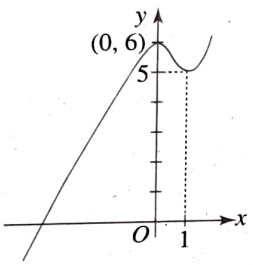
$\therefore \quad f'(x) < 0$

i.e., $f(x)$ is decreasing function.

2. (d) $f(x) = 2x^3 - 3x^2 + 6$

$f'(x) = 6x^2 - 6x = 6(x^2 - x) = 0$

$\Rightarrow \quad x = 0$ or $x = 1$



For inverse to exist function must be one one onto

Hence domain is $[1, \infty)$

Hence $a \geq 1$

3. (c) $f(x) = 1 - x - x^3$

$\Rightarrow \quad f'(x) = -1 - 3x^2 < 0$ for all x

$\Rightarrow \quad f(x)$ is decreasing for all x

Now $1 - f(x) - f^3(x) > f(1 - 5x)$

$\Rightarrow \quad f(f(x)) > f(1 - 5x)$

$\Rightarrow \quad f(x) < 1 - 5x$ (as $f(x)$ is decreasing)

$\Rightarrow \quad 1 - x - x^3 < 1 - 5x$

$\Rightarrow \quad x^3 - 4x > 0$

$\Rightarrow \quad x(x - 2)(x + 2) > 0$

$\Rightarrow \quad x \in (-2, 0) \cup (2, \infty)$

4. (b) Since $f''(x) > 0$

$\Rightarrow \quad f'(x)$ is always increasing.

$g'(x) = 2f'(2x^3 - 3x^2) \times (6x^2 - 6x) + f'(6x^2 - 4x^3 - 3)(12x - 12x^2)$

$= 12(x^2 - x) \cdot (f'(2x^3 - 3x^2) - f'(6x^2 - 4x^3 - 3))$

$= 12x(x - 1)\,[f'(2x^3 - 3x^2) - f'(6x^2 - 4x^3 - 3)]$

For increasing $g'(x) > 0$

Case I

$x < 0$ or $x > 1 \Rightarrow f'(2x^3 - 3x^2) > f'(6x^2 - 4x^3 - 3)$

$\Rightarrow \quad 2x^3 - 3x^2 > 6x^2 - 4x^3 - 3$ $\{\because f'(x)$ is increasing$\}$

$\Rightarrow \quad (x - 1)^2 \left(x + \frac{1}{2}\right) > 0$

$\Rightarrow \quad x > -\frac{1}{2} \quad \therefore x \in \left(-\frac{1}{2}, 0\right) \cup (1, \infty)$

Case II

$0 < x < 1$

$\Rightarrow \quad (x - 1)^2 (x + 1/2) < 0$

$\Rightarrow \quad x < -1/2$

So, no solution for this case.

5. (b) $f(x) = \sin 2x - 8(a + 1) \sin x + (4a^2 + 8a - 14)x$

$\therefore \quad f'(x) = 2 \cos 2x - 8(a + 1) \cos x + 4a^2 + 8a - 14$

$= 4 \cos^2 x - 2 - 8a \cos x - 8 \cos x + 4a^2 + 8a - 14$

Now $f(x)$ increases for all $x \in R$

$\therefore \quad f'(x) > 0$

$\cos^2 x - 2(a + 1) \cos x + (a + 1)^2 - 5 > 0$

$\Rightarrow \quad (\cos x - (a + 1))^2 > 5$

$\Rightarrow \quad \cos x - (a + 1) > \sqrt{5}$ or $\cos x - (a + 1) < -\sqrt{5}$

$\Rightarrow \quad a < -\sqrt{5} - 1 + \cos x \qquad a > \sqrt{5} - 1 + \cos x$

$\Rightarrow \quad a < -\sqrt{5} - 2 \qquad$ or $a > \sqrt{5}$

6. (c) $f'(x) = 2e^x + ke^{-x} + (2k + 1)$

$= 2e^x + \frac{k}{e^x} + (2k + 1)$

$= \frac{2(e^x)^2 + k + (2k + 1)e^x}{e^x}$

Now $f(x)$ is monotonically increasing.

$\Rightarrow \quad f'(x) \geq 0$ i.e., $2y^2 + (2k + 1)\, y + k \geq 0$

where $y = e^x$

$\therefore 2y^2 + (2k + 1)y + k \geq 0$ for all positive value of y.

For this $k \geq 0$.

7. (a) $f(x) = 2 \cot x + (2a + 1) \log_e |\text{cosec } x| + (2 - a)x$

$\Rightarrow \quad f'(x) = -2 \text{ cosec}^2 x - (2a + 1)\cot x + (2 - a)$

$= -2 \cot^2 x - (2a + 1)\cot x - a$

$= (\cot x + a)(-2 \cot x - 1) \leq 0$ in $\left(0, \frac{\pi}{2}\right)$

$\therefore \quad \cot x + a \geq 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $a \geq 0$

Multiple Correct Answers Type

8. (a, d)

(a) True

If $f(x)$ is not constant on $[-1, 1]$

$\Rightarrow \quad f'(x) > 0$ at some point in $-1 < x < 1$

$\Rightarrow \quad f$ is increasing at that point and we can't have $f(-1) = f(1)$

(b) Not true

If f is bounded curve changes its concavity

If f is unbounded then curve crosses x-axis

(c) Not true

e.g. $f(x) = x^4$

(d) $f'(x) = 2x^2 + 2x - 12 = 2(x + 3)(x - 2) < 0$ for $x \in (-3, 2)$ where $f(x)$ decreases.

Also $f''(x) = 4x + 2 > 0$ for $x > -\frac{1}{2}$

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9. **(a, b, c, d)**

$f(x) = x + \log_e(1 + x^2)$

$\Rightarrow \quad f'(x) = 1 + \dfrac{2x}{1+x^2}$

$= \dfrac{(x+1)^2}{1+x^2} \geq 0, \forall\, x \in R$

So, $f(x)$ is injective and at $x = -1$, tangent is not parallel to any of the chords.

Also $\lim\limits_{x \to -\infty} f(x)$

$= \lim\limits_{x \to -\infty} \log_e(e^x(1 + x^2))$

$= \log_e\left(\lim\limits_{x \to -\infty} \dfrac{1+x^2}{e^{-x}}\right)$

$= \log_e\left(\lim\limits_{x \to -\infty} \dfrac{2x}{-e^{-x}}\right)$

$= \log_e\left(\lim\limits_{x \to -\infty} \dfrac{2}{e^{-x}}\right)$

$= -\infty$

$\lim\limits_{x \to \infty} f(x) = \infty$

Thus $f(x)$ has range R, so $f(x)$ is surjective.

10. **(b, c, d)**

$f(x) = x - \dfrac{1}{x}$

$f'(x) = 1 + \dfrac{1}{x^2} > 0$

$\Rightarrow$ f is increasing

$f(1) = f(-1) = 0$

$\Rightarrow$ $f(x)$ is many–one

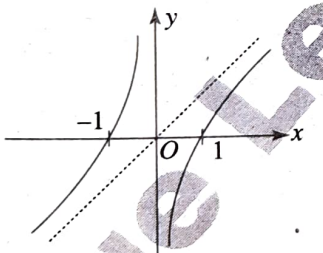
When $x \to 0^+$; $f \to -\infty$

When $x \to \infty$, $f \to \infty$

When $x \to 0^-$; $f \to +\infty$

And when $x \to -\infty$, $f \to -\infty$

So the graph of the function is as shown in the figure.



From the graph any line $y = k$ meets the curve at two distinct points.

Also $f''(x) = -2/x^3$

$f''(x) > 0$ for $x < 0$

and $f''(x) < 0$ for $x > 0$

Hence $x = 0$ point of inflection.

11. **(a, d)**

As $f(x)$ is even, $f(-2) = f(2)$

As $f(x)$ is increasing in $[2, 6]$, $f(3) < f(5) = f(-5)$

Also $f(3) < f(5) \Rightarrow f(-3) < f(5)$

12. **(b, d)**

If $f(x) = \begin{cases} -e^{-x} + k, & x \leq 0 \\ e^x + 1, & 0 < x < 1 \\ ex^2 + \lambda, & x \geq 1 \end{cases}$

$y = -e^{-x} + k$ is increasing for $x < 0$

$y = e^x + 1$ is increasing for $0 < x < 1$

$y = ex^2 + \lambda$ is increasing for $x > 1$

$\therefore$ $f(x)$ is increasing for $\forall\, x \in R$

$\Rightarrow$ $f(0^-) \leq f(0^+)$

$\Rightarrow$ $-1 + k \leq 1 + 1$

$\Rightarrow$ $k \leq 3$

and $f(1^-) \leq f(1^+)$

$\Rightarrow$ $e + 1 \leq e + \lambda$

$\Rightarrow$ $\lambda \geq 1$

DPP 6.2

Single Correct Answer Type

1. (d) For $x_1, x_2 \in \left(0, \dfrac{\pi}{2}\right)$, we know $\dfrac{x}{\tan x}$ is decreasing function

$\therefore$ for $x_1 < x_2 \Rightarrow \dfrac{x_1}{\tan x_1} > \dfrac{x_2}{\tan x_2}$

$\Rightarrow \dfrac{\tan x_2}{\tan x_1} > \dfrac{x_2}{x_1}$

2. (b) Given $f''(x) - 3f'(x) > 3$

$\Rightarrow \dfrac{d}{dx}(e^{-3x} f'(x)) > 3e^{-3x}$

$\Rightarrow \dfrac{d}{dx}(e^{-3x} f'(x) + e^{-3x}) > 0$

$\Rightarrow$ $e^{-3x}(f'(x) + 1)$ is increasing function

Also $e^{-3x}(f'(x) + 1) > f'(0) + 1\ \forall\, x > 0$

$\Rightarrow$ $f'(x) + 1 > 0$

$\Rightarrow$ $f(x) + x$ is an increasing function.

3. (b) We have $(x - 41)^{49} + (x - 49)^{41} + (x - 2009)^{2009} = 0$

Let $f(x) = (x - 41)^{49} + (x - 49)^{41} + (x - 2009)^{2009}$

$\therefore$ $f'(x) = 49(x - 41)^{48} + 41(x - 49)^{40} + 2009\,(x - 2009)^{48} > 0$

Hence $f(x)$ will cut x-axis only once $\Rightarrow$ 1 real root

4. (d) Given $g(x) = \log_e(h(x))$

$\therefore$ $g'(x) = \dfrac{h'(x)}{h(x)}$

$\therefore$ $g''(x) = \dfrac{h(x)h''(x) - (h'(x))^2}{h^2(x)} < 0$ (given)

$\therefore$ $g''(x) < 0 \Rightarrow g(x)$ is concave down

5. (b) $f(x) = \sin x + x$,

$\Rightarrow$ $f'(x) = \cos x + 1 > 0$

$\Rightarrow$ $f(x)$ is increasing function

Also $f''(x) = -\sin x < 0$

$\therefore$ f is concave downward for $x \in \left[0, \dfrac{\pi}{2}\right]$

Now $g(x) = |k|x^2$ is concave upward and increasing

So, if $g\left(\dfrac{\pi}{2}\right) \leq f\left(\dfrac{\pi}{2}\right)$, then $f(x) \geq g(x)$

$\Rightarrow$ $1 + \dfrac{\pi}{2} \geq |k| \dfrac{\pi^2}{4}$

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$$\Rightarrow \qquad k \in \left[\frac{-(2\pi+4)}{\pi^2}, \frac{(2\pi+4)}{\pi^2}\right]$$

6. (b) Let $f(x) = 4x + 8\cos x + \tan x - 2\sec x - 4\log\{\cos x(1+\sin x)\} - 6$

Now $f(0) = 0 + 8 + 0 - 2 - 4\log 1 - 6 = 0$ and

$$f'(x) = 4 - 8\sin x + \sec^2 x - 2\sec x\tan x + 4\tan x - \frac{4\cos x}{1+\sin x}$$

$$= 4(1-2\sin x) + \sec^2 x(1-2\sin x) + \frac{4\sin x}{\cos x} - \frac{4\cos x[1-\sin x]}{\cos^2 x}$$

$$= (4+\sec^2 x)(1-2\sin x) + \frac{4}{\cos x}(\sin x - 1 + \sin x)$$

$= (1-2\sin x)(4+\sec^2 x - 4\sec x)$

$= (1-2\sin x)(\sec x - 2)^2$

Thus $f'(x) > 0$ if $1 - 2\sin x > 0$ i.e. $x \in (0, \pi/6)$.

Therefore f increases on $[0, \pi/6)$.

But $f(0) = 0$, so $f(x) \geq f(0)$ for all $x \in [0, \pi/6)$

Thus the largest such value of λ is $\pi/6$.

Comprehension Type

7. (b) Consider $g(x) = \dfrac{f(x)}{x}$

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{f'(x)}{x^2}\left(x - \frac{f(x)}{f'(x)}\right)$$

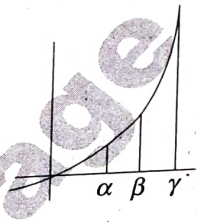
as $\dfrac{f'(x)}{x^2} > 0$

Let $h(x) = x - \dfrac{f(x)}{f'(x)}$

$$\therefore \qquad h'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2}$$

$$= \frac{f(x)f''(x)}{f'(x)^2} > 0$$

$\therefore$ $x - \dfrac{f(x)}{f'(x)}$ is increasing function.



$$x - \frac{f(x)}{f'(x)} > 0 - \frac{f(0)}{f'(0)} = 0$$

$\therefore$ $g'(x) > 0$.

$$\therefore \qquad \frac{f(\gamma)}{\gamma} > \frac{f(\beta)}{\beta} > \frac{f(\alpha)}{\alpha}$$

$\therefore$ B is false

8. (b) As the function is concave upward

$$\frac{f(\alpha)+f(\beta)}{2} > f\left(\frac{\alpha+\beta}{2}\right)$$

9. (a) As $\gamma + \beta - \alpha > \gamma$

$$\frac{f(\gamma+\beta-\alpha)}{\gamma+\beta-\alpha} > \frac{f(\gamma)}{\gamma}$$

10. (c) $f''(x) > 0 \Rightarrow f'(x)$ is an increasing function

$h'(x) = \sin 2x\,(f'(\sin^2 x) - f'(\cos^2 x))$

$f'(x) = 0 \Rightarrow \sin^2 x = 0 \Rightarrow x = 0$

or $f'(\sin^2 x) = f'(\cos^2 x) \Rightarrow \sin^2 x = \cos^2 x \Rightarrow \tan^2 x = 1 \Rightarrow x = \pm\dfrac{\pi}{4}$

11. (a) $f'(\sin^2 x) < f'(\cos^2 x) \Rightarrow \sin^2 x < \cos^2 x \Rightarrow \tan^2 x < 1$

$\Rightarrow x \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

12. (b) $h(x)$ is increasing $\Rightarrow h'(x) > 0$

Case I

(i) $\sin 2x > 0 \Rightarrow x \in \left(0, \dfrac{\pi}{2}\right)$

(ii) $f'(\sin^2 x) > f'(\cos^2 x) \Rightarrow \tan^2 x > 1$

$\Rightarrow x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

Case II

(i) $\sin 2x < 0 \Rightarrow x \in \left(-\dfrac{\pi}{2}, 0\right)$

(ii) $f'(\sin^2 x) > f'(\cos^2 x) \Rightarrow \tan^2 x > 1$

$\Rightarrow x \in \left(\dfrac{-\pi}{2}, -\dfrac{\pi}{4}\right)$

DPP 6.3

Single Correct Answer Type

1. (d) $f(x) = \tan x + \cot x + \cos x$

$$= \frac{2}{\sin 2x} + \cos x$$

Both $\dfrac{2}{\sin 2x}$ and $\cos x$ decreasing on $[\pi/6, \pi/4]$ and thus the greatest value occurs at $x = \pi/6$

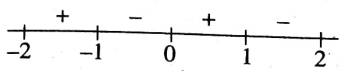
i.e. $\dfrac{2}{\sin \pi/3} + \cos \pi/6 = \dfrac{4}{\sqrt{3}} + \dfrac{\sqrt{3}}{2} = \dfrac{11}{2\sqrt{3}} = \dfrac{11\sqrt{3}}{6}$

2. (b) $f(x) = \begin{cases} (x+1)^3 & -2 < x \leq -1 \\ x^{2/3} - 1 & -1 < x \leq 1 \\ -(x-1)^2 & 1 < x < 2 \end{cases}$

Clearly $f(x)$ is continuous.

$$\Rightarrow \qquad f'(x) = \begin{cases} 3(x+1)^2 & -2 < x < -1 \\ \dfrac{2}{3} \times x^{-1/3} & -1 < x < 1 - \{0\} \\ -2(x-1) & 1 < x < 2 \end{cases}$$

Sign scheme of $f'(x)$



Critical points $-1, 0, 1$

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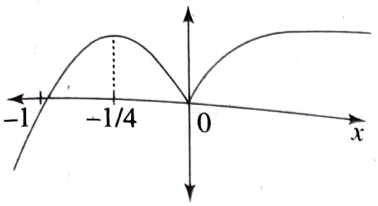
3. (d) $f(x) = \begin{cases} x+\sqrt{x} & \text{if } x \geq 0 \\ x+\sqrt{-x} & \text{if } x < 0 \end{cases}$

The graph of $f(x)$ is shown with $f'(x) = 0$ as $x = -1/4$. Also derivative fails at $x = 0$. Hence there are two critical points.



4. (d) $f(x) = \dfrac{\tan\left(x+\frac{\pi}{6}\right)}{\tan x} = \cot x \tan\left(x+\frac{\pi}{6}\right)$

$$f'(x) = \cot x \sec^2\left(x+\frac{\pi}{6}\right) - \operatorname{cosec}^2 x \tan\left(x+\frac{\pi}{6}\right)$$

$$\therefore \quad f''(x) = 2\cot x \sec^2\left(x+\frac{\pi}{6}\right)\tan\left(x+\frac{\pi}{6}\right) - \operatorname{cosec}^2 x \sec^2\left(x+\frac{\pi}{6}\right) - \operatorname{cosec}^2 x \sec^2\left(x+\frac{\pi}{6}\right) + 2\operatorname{cosec}^2 x \cot x \tan\left(x+\frac{\pi}{6}\right)$$

Now $f'(x) = 0$

$$\Rightarrow \quad \frac{1}{2}\sin 2x = \frac{1}{2}\sin\left(2x+\frac{\pi}{3}\right)$$

$$\Rightarrow \quad 2x = \pi - 2x - \frac{\pi}{3}$$

$$\Rightarrow \quad x = \frac{\pi}{6}$$

Then $f''(\pi/6) > 0$

$\therefore$ At $x = \dfrac{\pi}{6}$, $f(x)$ is minimum and there is no other minimum in $\left(0, \dfrac{\pi}{2}\right)$.

$\therefore$ The minimum value of $f(x) = f\left(\dfrac{\pi}{6}\right) = \dfrac{\tan\left(\frac{\pi}{6}+\frac{\pi}{6}\right)}{\tan\frac{\pi}{6}} = \dfrac{\sqrt{3}}{\frac{1}{\sqrt{3}}} = 3$

5. (b) $f(x) = \begin{cases} x^2+2, & 1 \leq x < 2 \\ \dfrac{x^2+2}{2}, & 2 \leq x < 3 \\ \dfrac{x^2+2}{3} & x = 3 \end{cases}$

$\therefore$ Least value of $f(x)$ in $[1, 2]$ is 3
Least value of $f(x)$ in $[2, 3]$ is 3

$$f(3) = \frac{11}{3}$$

$\therefore$ Least value of $f(x)$ is 3

6. (c) Local maximum at $x = 2$

$$\Rightarrow \quad \lim_{h\to 0} f(2+h) \leq f(2)$$

$$\Rightarrow \quad \lim_{h\to 0}(\sqrt{a+14} - |2+h-48|) \leq 3 - 2^2$$

$$\Rightarrow \quad \sqrt{a+14} \leq 45 \Rightarrow a \leq 2011$$

7. (d) $f'(x) = 5x^4 - 20x^3 + 15x^2$

$$f''(x) = 20x^3 - 60x^2 + 30x$$

$$f'(x) = 0 \Rightarrow x^2(x^2 - 4x + 3) = 0$$

$$\Rightarrow \quad x^2(x-1)(x-3) = 0 \Rightarrow x = 0, 1, 3$$

$f''(1) < 0$ and $f''(3) = 540 - 540 + 90 > 0$

and $f''(0) = 0$ and $f'''(0) \neq 0$

$\therefore$ Maximum at $x = 1$ and minimum at $x = 3$

8. (c) $f(x) = |x-1| + |x+4| + |x-9| + \ldots + |x-2500|$ is non-differentiable at $x = 1^2$, $x = 2^2$, ..., $x = 50^2$.
$f(x)$ is minimum at the middle term of the above series which are $x = 25^2$ and $x = 26^2$.

9. (b) The slope of the tangent to the curve

$$y = 2e^x \sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right) = e^x \cos x$$

$$S = \frac{dy}{dx} = e^x(-\sin x + \cos x)$$

Now, $\dfrac{dS}{dx} = e^x(-\sin x + \cos x - \cos x - \sin x)$

$$= -2e^x \sin x$$

$$\frac{dS}{dx} = 0 \Rightarrow -2e^x \sin x = 0$$

$$\Rightarrow \quad x = 0, \pi, 2\pi \ (\because 0 \leq x \leq 2\pi)$$

Value of S at $x = 0$ is 1; value of S at $x = \pi$ is $-e^{\pi}$
Value of S at $x = 2\pi$ is $e^{2\pi}$.
$\therefore$ S is minimum at $x = \pi$.

10. (b) $f(x) = x^3 - 3ax^2 + 3(a^2-1)x + 1$

$$f'(x) = 3x^2 - 6ax + 3(a^2-1)$$

$$= 3(x-(a+1))(x-(a-1))$$

so $f'(x) = 0 \Rightarrow x = a+1$ or $a-1$

$\therefore$ $a+1 \in (-2, 4)$ if $a \in (-3, 3)$

and $a-1 \in (-2, 4)$ if $a \in (-1, 5)$

$\therefore$ $a \in (-1, 3)$

Multiple Correct Answers Type

11. (a, d)

$f(x) = ax\,e^{-bx}$ has a local maximum at the point $(2, 10)$

$\therefore$ $f(2) = 10$, $2ae^{-2b} = 10$

$\Rightarrow$ $ae^{-2b} = 5$ (i)

$f'(x) = a[e^{-bx} - bx\,e^{-bx}]$

$f'(2) = 0$

$\Rightarrow$ $a(e^{-2b} - 2be^{-2b}) = 0$

$\Rightarrow$ $ae^{-2b}(1-2b) = 0$

$\Rightarrow$ $b = 1/2$

From (i) if $b = 1/2$, then

$a = 5e$ or $a = 0$ (not possible)

$\therefore$ $a = 5e$ and $b = 1/2$

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12. (b, d)

$$f(x)=\frac{e^x}{1+x^2}$$

$$g(x)=f'(x)=\frac{e^x(x-1)^2}{(1+x^2)^2}$$

$$\Rightarrow \quad g'(x)=\frac{(x-1)(x^3-3x^2+5x+1)e^x}{(x^2+1)^3}$$

Now, $h(x)=x^3-3x^2+5x+1$

$\Rightarrow \quad h'(x)=3x^2-6x+5$ which is always positive.

Hence $x^3-3x^2+5x+1=0$ has only one real root which is not $x=1$.

Also $h(0)=1$, $h(-1)=-8$. So $h(x)=0$ has one negative root $x=\alpha\in(-1,0)$.

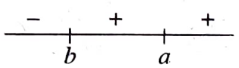
Now from the sign scheme of $g'(x)$,

$x=\alpha$ is the point of maxima and $x=1$ is the point of minima.

13. (b, d)

Given $f'(x)=(x-a)^{2010}(x-b)^{2009}$ and $a>b$

Sign scheme of $f'(x)$ is



From sign scheme, $x=b$ is point of minima and $x=a$ is neither maxima nor minima.

14. (a, d)

We have $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a}[f(x)]$.

This can occur only when $\lim\limits_{x\to a} f(x)$ is an integer.

$\Rightarrow \quad f(a^+)>f(a)$ and $f(a^-)>f(a)$

$\Rightarrow \quad x=a$ must be point of local minima.

15. (a, b, c)

$$f(x)=\log_e(\sqrt{1-x^2}-x)$$

$\sqrt{1-x^2}-x$ is defined for $x\in[-1,1]$

If $x\in[-1,0]$, $\sqrt{1-x^2}-x>0$

If $x>0$, then $\sqrt{1-x^2}>x\Rightarrow 1-x^2>x^2$

i.e. $x\in\left(0,\frac{1}{\sqrt{2}}\right)$

Domain of f is $\left[-1,\frac{1}{\sqrt{2}}\right]$

$$f'(x)=\frac{\frac{-x}{\sqrt{1-x^2}}-1}{\sqrt{1-x^2}-x}$$

$f'(x)>0$ if $\frac{-x}{\sqrt{1-x^2}}-1>0$

or $\quad -x-\sqrt{1-x^2}>0$

or $\quad x\in\left[-1,\frac{-1}{\sqrt{2}}\right)$

$\therefore f(x)$ increases in $\left[-1,-\frac{1}{\sqrt{2}}\right]$ and decreases in $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$\therefore f$ has local max. at $x=-\frac{1}{\sqrt{2}}$

Also $\lim\limits_{x\to\frac{1}{\sqrt{2}}^-}\log_e(\sqrt{1-x^2}-x)=-\infty$

$\therefore \quad f$ has no minima.

DPP 6.4

Single Correct Answer Type

1. (d) $f(x)=\begin{cases}x^3(1-x), & x\le 0\\ x\ln x+3x, & x>0\end{cases}$

$f(0)=0$ (using $x^3(1-x)$)

$f(0^-)=0$

$$f(0^+)=\lim_{x\to 0}(x\ln x+3x)$$

$$=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}+0$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$=0$$

Thus, function is continuous at $x=0$.

$f(0)>f(0^-)$, $f(0)>f(0^+)$

$\therefore x=0$ is point of maxima.

Also $(x\ln x+3x)'=4+\ln x$

$4+\ln x=0\Rightarrow x=e^{-4}$

$(x\ln x+3x)''=1/x>0$ for $x>0$

Thus, $x=e^{-4}$ is point of minima

$$\lim_{x\to-\infty}x^3(1-x)=-\infty$$

and $\quad \lim\limits_{x\to\infty}(x\ln x+3x)=\infty$

$\therefore$ Range of $f(x)$ is R.

2. (a) The ordinate of any point on the curve is given by $y=\frac{x^3}{(x-a)^2}$

$$\frac{dy}{dx}=\frac{3x^2}{(x-a)^2}-\frac{2x^3}{(x-a)^3}=\frac{x^2(x-3a)}{(x-a)^3}$$

$\frac{dy}{dx}=0\Rightarrow x=0$ or $x=3a$

From sign scheme of $x=3a$ is point of minima and value of y at $x=3a$ is

$$y=\frac{(3a)^3}{(2a)^2}=\frac{27a^3}{4a^2}=\frac{27}{4}a$$

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3. (a) Let $y = x - x^p$, where x is the fraction

$$\Rightarrow \quad \frac{dy}{dx} = 1 - px^{p-1}$$

For maximum or minimum, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 1 - px^{p-1} = 0 \Rightarrow x = \left(\frac{1}{p}\right)^{1/(p-1)}$$

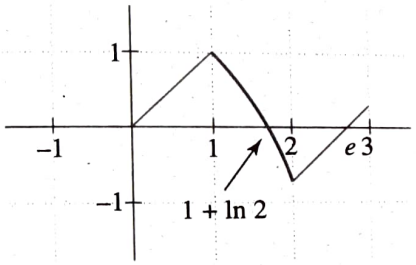
Now, $\frac{d^2y}{dx^2} = -p(p-1)\,x^{p-2}$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\left(\frac{1}{p}\right)^{1/(p-1)}} = -p\left(\frac{1}{p}\right)^{(p-2)/(p-1)} < 0$$

$\therefore y$ is maximum at $x = \left(\frac{1}{p}\right)^{1/(p-1)}$

4. (c) $f(x) = g'(x) = \begin{cases} 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$

$g'(x) = 0 \Rightarrow e^{x-1} = 2 \Rightarrow x = 1 + \ln 2$
and $x - e = 0 \Rightarrow x = e$
Graph of $f(x)$ is as shown in the following figure.



From the figure, derivative changes sign from '+' to '−' at $x = 1 + \ln 2$ and from '−' to '+' at $x = e$
Hence $x = 1 + \ln 2$ is point of maxima and $x = e$ is point of minima

5. (b) $y^2 - xy = \left(y - \frac{x}{2}\right)^2 - \frac{x^2}{4} \Rightarrow y^2 - xy$ is decreasing for $y \le \frac{x}{2}$

and increasing for $y \ge \frac{x}{2}$

Thus the largest value of $y^2 - xy$ must be at $y = 0$, $\frac{x}{2}$ or 1.

The values are $0, \frac{x^2}{4}, 1 - x$ for $x \in (0, 1)$

And for $x \in (0, 1)$ $g(x) = \max\left(\frac{x^2}{4}, 1 - x\right)$

Also $x^2 + 4x - 4 = 0 \Rightarrow x = -2 \pm \sqrt{8}$

$\Rightarrow$ $g(x) = 1 - x$ for $x \le \sqrt{8} - 2$ and $\frac{x^2}{4}$ for $x \ge \sqrt{8} - 2$

$\Rightarrow$ $g(x)$ is minimum at $x = \sqrt{8} - 2$ and minimum value is $3 - \sqrt{8}$

6. (d) Let $a < b$ and $f(x) = |x - a| + |x - b|$ $\forall x \in R$
So, $f(x)$ decreasing in $(-\infty, a]$ constant in $[a, b]$ and increasing in $[b, \infty)$.

$$|a - 1| + |b - 1| = |a| + |b| = |a + 1| + |b + 1|$$
$$\Rightarrow \quad f(0) = f(1) = f(-1)$$
$$\Rightarrow \quad \{-1, 0, 1\} \in \{a, b\}$$
$$\therefore \quad |a - b|_{\min} = 2.$$

7. (c) Let $f(x) = 2x^3 - 6x + 2\sin a + 3$

$$\Rightarrow \quad f'(x) = 6x^2 - 6 = 0 \Rightarrow x = \pm 1$$

Since equation $f(x) = 0$ has only one real root

$$f(-1).f(1) > 0$$
$$\Rightarrow \quad (2\sin a + 7)(2\sin a - 1) > 0$$
$$\Rightarrow \quad \sin a > \frac{1}{2} \Rightarrow a \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$$

8. (c) $\frac{2f(x)\cdot f'(x)}{\sqrt{1 - (f(x))^4}} - 2x \ge 0$

$$\Rightarrow \quad \frac{d}{dx}\left(\sin^{-1}(f(x))^2 - x^2\right) \ge 0$$

Then $g(x) = \sin^{-1}((f(x))^2) - x^2$ is a non–decreasing function.

$$\Rightarrow \quad \lim_{x \to x_1^+} g(x) \le \lim_{x \to x_2^-} g(x)$$

$$\Rightarrow \quad \frac{\pi}{2} - x_1^2 \le \frac{\pi}{6} - x_2^2$$

$$\Rightarrow \quad x_1^2 - x_2^2 \ge \frac{\pi}{3}$$

9. (b) $ab = 2a + 3b \Rightarrow b = \frac{2a}{a-3}$

Now $z = ab = \frac{2a^2}{a-3}$

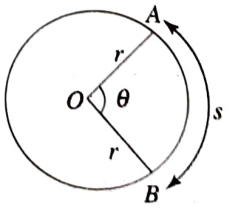
$$\Rightarrow \quad \frac{dz}{da} = \frac{2[(a-3)2a - a^2]}{(a-3)^2} = \frac{2[a^2 - 6a]}{(a-3)^2}$$

Put $\frac{dz}{da} = 0$, $\therefore a^2 - 6a = 0$, $a = 0, 6$

Clearly $a = 6$ is point of minima
When $a = 6$, $b = 4 \Rightarrow (ab)_{\text{min.}} = 6 \times 4 = 24$.

10. (b) $a + b + c + d + e + f + g + h = 8$
Let $a + b + c + d = x$
$\therefore e + f + g + h = 8 - x$
Let $y = x^2 + (8 - x)^2$
$\therefore y = 2x^2 - 16x + 64$
$= 2[x^2 - 8x + 32]$
$= 2[(x - 4)^2 + 16]$
$\therefore y_{\min} = 32$ when $x = 4$

11. (d)



Perimeter of a sector = p. Let AOB be the sector with radius r.
If angle of the sector be θ radians, then area of sector

$$(A) = \frac{1}{2}r^2\theta \qquad \text{(i)}$$

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Length of arc$(s) = r\theta$ or $\theta = \frac{s}{r}$.

Therefore perimeter of the sector

$$p = r + s + r = 2r + s \qquad \text{(ii)}$$

Substituting $\theta = \frac{s}{r}$ in (i), $A = \left(\frac{1}{2}r^2\right)\left(\frac{s}{r}\right) = \frac{1}{2}rs$

$\Rightarrow s = \frac{2A}{r}$. Now substituting the value of s in (ii), we get

$$p = 2r + \left(\frac{2A}{r}\right) \text{ or } 2A = pr - 2r^2.$$

Differentiating with respect to r, we get

$$2\frac{dA}{dr} = p - 4r$$

For maximum value of area

$$\frac{dA}{dr} = 0 \text{ or } p - 4r = 0 \text{ or } r = \frac{p}{4}$$

12. (b) Given equation is $e^x = kx^2$

$$\Rightarrow \qquad k = \frac{e^x}{x^2}$$

Now let $f(x) = \frac{e^x}{x^2}$,

$$\therefore \qquad f'(x) = \frac{(x-2)e^x}{x^3}$$

$f'(x) = 0 \Rightarrow x = 2$, which is point of minima.

Also $f(2) = \frac{e^2}{4}$

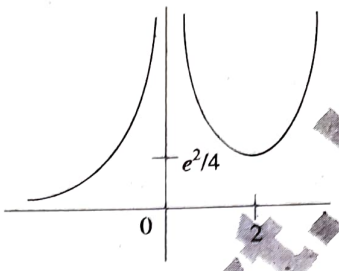
$$\lim_{x\to\infty} \frac{e^x}{x^2} = \infty \qquad \text{(using L'Hopital Rule twice)}$$

$$\lim_{x\to-\infty} \frac{e^x}{x^2} = 0$$

$$\lim_{x\to 0} \frac{e^x}{x^2} = \infty$$

Further $f(x) > 0, \forall\, x \in R$.

From this information, graph of $f(x)$ is as shown below.



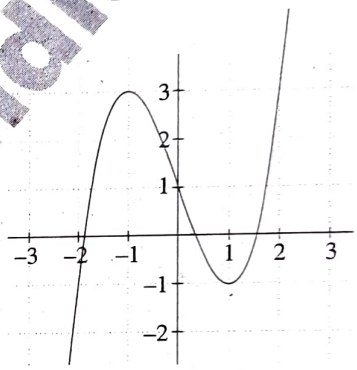
Thus, three real distinct solutions $k > \frac{e^2}{4}$, $k \in I$.

So, $k_{min} = 2$.

13. (d) $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$f(-1) = 3, f(1) = -1; f(3) = 19 > 0$

Graph of $f(x)$ is as shown in the following figure.



$f(x) = 0$ has root x_1, x_2, x_3 such that

$x_1 < -1, -1 < x_2 < 1, 1 < x_3 < 3$

$\Rightarrow$ $f(x) = x_1$ has only one real root since, $x_1 < -1$

$f(x) = x_2$ has 3 real roots since $-1 < x_2 < 1$

$f(x) = x_3$ has 3 real roots since $1 < x_3 < 3$

The roots of $f(f(x)) = 0$ only if $f(x) = x_1, f(x) = x_2, f(x) = x_3$

$\therefore$ Number real roots of $f(f(x)) = 0$ is 7

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CHAPTER 7

DPP 7.1

Single Correct Answer Type

1. (d) $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx$

$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2} dx}{1 - 2\left(2\cos^2\left(\frac{3x}{2}\right) - 1\right)}$

$= \int \frac{2\cos\frac{x}{2}\left(4\cos^3\frac{3x}{2} - 3\cos\frac{3x}{2}\right)}{3 - 4\cos^2\frac{3x}{2}} dx$

$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2} dx = -\int \cos 2x\, dx - \int \cos x\, dx$

$= -\left(\frac{\sin 2x}{2} + \sin x\right) + C$

2. (a) $I = \int \frac{\sec x \operatorname{cosec} x}{2\cot x - \sec x \operatorname{cosec} x} dx$

$= \int \frac{dx}{2\cos^2 x - 1}$

$= \int \sec 2x\, dx$

$= \frac{1}{2}\ln|\sec 2x + \tan 2x| + C$

3. (c) $\int \frac{1}{x}\ln\left(\frac{x}{e^x}\right) dx$

$= \int \frac{1}{x}(\ln x - \ln e^x) dx$

$= \int \frac{\ln x - x}{x} dx$

$= \left[\int \frac{1}{x}\ln x\, dx - \int \frac{1}{x} x\, dx\right]$

$= \frac{1}{2}(\ln x)^2 - x + C$

4. (c) $\int \frac{\cos^{n-1} x}{\sin^{n+1} x} dx = \int \cot^{n-1} x \operatorname{cosec}^2 x\, dx$

$= \frac{-\cot^n x}{n} + C$

5. (c) Differentiating both sides gives

$x^{26} \cdot (x-1)^{17}(5x-3) = \frac{1}{k}[x^{27} \cdot 18(x-1)^{17} + (x-1)^{18} 27x^{26}]$

$= \frac{x^{26}(x-1)^{17}}{k}[18x + 27(x-1)]$

$= \frac{x^{26}(x-1)^{17}}{k}(45x - 27)$

$= 9\frac{x^{26}(x-1)^{17}}{k}(5x - 3)$

$\Rightarrow \quad k = 9$

6. (d) $\int \frac{x + (\cos^{-1} 3x)^2}{\sqrt{1-9x^2}} dx = \int \frac{x}{\sqrt{1-9x^2}} dx + \int \frac{(\cos^{-1} 3x)^2}{\sqrt{1-9x^2}} dx$

$= -\frac{1}{18} 2\sqrt{1-9x^2} - \frac{1}{9}(\cos^{-1} 3x)^3 + C$

$\therefore \quad A = -\frac{1}{9}, B = -\frac{1}{9}$

7. (c) Let $I_9 = \int \tan^9 x\, dx = \int \tan^7 x(\sec^2 x - 1) dx$

i.e., $I_9 = \frac{\tan^8 x}{8} - I_7$

Similarly, $I_7 = \frac{\tan^6 x}{6} - I_5$

.....

.....

.....

$I_3 = \frac{\tan^2 x}{2} - I_1 = \frac{\tan^2 x}{2} - \int \tan x\, dx$

$= \frac{\tan^2 x}{2} + \log|\cos x|$

$\therefore I_9 =$ (a polynomial of degree 8 in $\tan x$) + $\log|\cos x| + C$

8. (b) $I = \int \frac{(e^x + \cos x + 1) - (e^x + \sin x + x)}{e^x + \sin x + x} dx$

$= \log_e(e^x + \sin x + x) - x + C$

$\therefore \quad f(x) = e^x + \sin x + x$ and $g(x) = -x$

$f(x) + g(x) = e^x + \sin x$

9. (b) Put $x = u^6$

$\therefore \quad I = 6\int \frac{u^5 + u^3 + 1}{1 + u^2} du$

$= 6\int u^3\, du + 6\int \frac{du}{1 + u^2}$

$= \frac{3u^4}{2} + 6\tan^{-1} u + C$

$= \frac{3x^{\frac{2}{3}}}{2} + 6\tan^{-1}\left(\frac{1}{x^{\frac{1}{6}}}\right) + C$

10. (a) $\int \frac{e^x(x-2)}{x(x^2 + e^x)} dx = \int \frac{3x^2 + e^x + xe^x - 3x^2 - 3e^x}{x(x^2 + e^x)} dx$

$= \int \frac{d(x^3 + xe^x)}{x^3 + xe^x} - 3\int \frac{dx}{x}$

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$$= \ln(x^3 + xe^x) - 3\ln x + c$$

$$= \ln\left(\frac{x^2 + e^x}{x^2}\right) + c = \ln\left(1 + \frac{e^x}{x^2}\right) + c$$

DPP 7.2

Single Correct Answer Type

1. (b) $I = \frac{1}{2}\int 2x\sqrt{\frac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}}\,dx$

$x^2 + 1 = t \Rightarrow 2x\,dx = dt$

$$I = \frac{1}{2}\int\sqrt{\frac{2\sin t - \sin 2t}{2\sin t + \sin 2t}}\,dt$$

$$= \frac{1}{2}\int\sqrt{\frac{2 - 2\cos t}{2 + 2\cos t}}\,dt$$

$$= \frac{1}{2}\int \tan\frac{t}{2}\,dt$$

$$= \frac{1}{2}\frac{\ln\left|\sec\frac{t}{2}\right|}{\frac{1}{2}} + c$$

$$= \ln\left|\sec\left(\frac{x^2+1}{2}\right)\right| + c$$

2. (c) $I = \int\frac{\operatorname{cosec} x}{\cos^2\left(1 + \log\tan\frac{x}{2}\right)}\,dx$

Let $1 + \log\tan\frac{x}{2} = t$

$\therefore \quad \frac{1}{\sin x}\,dx = dt$

$\therefore \quad I = \int\frac{dt}{\cos^2 t}$

$= \tan t + c$

$= \tan\left(1 + \log\tan\frac{x}{2}\right) + c$

3. (b) $\int\frac{dx}{x\sqrt{x^6 - 16}} = \int\frac{x^2\,dx}{x^3\sqrt{x^6 - 16}}$

$$= \frac{1}{3}\int\frac{dt}{t\sqrt{t^2 - 16}} \quad (\text{putting } x^3 = t)$$

$$= \frac{1}{48}\int\frac{dt}{\frac{t}{4}\sqrt{\frac{t^2}{16} - 1}}$$

$$= \frac{1}{12}\sec\frac{t}{4} + c$$

$$= \frac{1}{12}\sec\frac{x^3}{4} + c$$

4. (b) $\int\frac{\sin(4x - 2x)dx}{\sin(2x)\cos(2x)\cos(4x)}$

$$= \int\frac{\sin(4x)dx}{\sin(2x)\cos(4x)} - \int\sec 2x\,dx$$

$$= 2\int\frac{\cos 2x\,dx}{\cos 4x} - \frac{1}{2}(\log|\sec 2x - \tan 2x|)$$

$$= 2\int\frac{\cos 2x\,dx}{1 - 2\sin^2 2x} - \frac{1}{2}(\log|\sec 2x - \tan 2x|)$$

$$= \int\frac{dt}{1 - 2t^2} - \frac{1}{2}(\log|\sec 2x - \tan 2x|)$$

5. (c) $I = \int x\,2^{\ln(x^2+1)}\,dx$

Let $x^2 + 1 = t; \Rightarrow x\,dx = \frac{dt}{2}$

Hence $I = \frac{1}{2}\int 2^{\ln t}\,dt$

$$= \frac{1}{2}\int t^{\ln 2}\,dt$$

$$= \frac{1}{2}\cdot\frac{t^{\ln 2+1}}{\ln 2 + 1} + C$$

$$= +\frac{1}{2}\cdot\frac{(x^2+1)^{\ln 2+1}}{\ln 2 + 1}C$$

6. (d)

$$\int\frac{\sin x}{\sin\left(x - \frac{\pi}{4}\right)}\,dx$$

$$= \int\frac{\sin\left(u + \frac{\pi}{4}\right)}{\sin u}\,du \quad \left(\text{put } u = x - \frac{\pi}{4}\right)$$

$$= \int\frac{\sin u\cos\frac{\pi}{4} + \cos u\sin\frac{\pi}{4}}{\sin u}\,du$$

$$= \frac{1}{\sqrt{2}}u + \frac{1}{\sqrt{2}}\log|\sin u| + c$$

$$= \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\log(|\sin x - \cos x|) + c$$

$\therefore \quad A = \frac{1}{\sqrt{2}}, f(x) = x$

7. (c) $I = \int\frac{\cos x - x\sin x}{\sqrt{x\cos x}}\,dx$

Put $x\cos x = t^2$

$\Rightarrow \quad (\cos x - x\sin x)dx = 2t\,dt$

$\Rightarrow \quad I = \int\frac{2t\,dt}{t} = 2t + C = 2\sqrt{x\cos x} + C$

8. (b) $\int\frac{2x+1}{(x^2 + 4x + 1)^{3/2}}\,dx = \int\frac{2x+1}{x^3\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)^{3/2}}\,dx$

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$$= \int \frac{2x^{-2} + x^{-3}}{\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)^{3/2}} dx$$

Now put $\frac{1}{x^2} + \frac{4}{x} + 1 = t^2$

9. (b) $I = \int \frac{2x+3}{(x^2+3x)(x^2+3x+2)+1} dx$

Put $x^2 + 3x = t \Rightarrow (2x+3)dx = dt$

$\Rightarrow$

$I = \int \frac{dt}{t(t+2)+1}; \int \frac{dt}{(t+1)^2} = C - \frac{1}{t+1} = C - \frac{1}{x^2+3x+1}$

$\Rightarrow \quad f(1) = 5$

10. (b) $e^{\sqrt{\sin x}} = t \Rightarrow \frac{e^{\sqrt{\sin x}}}{2\sqrt{\sin x}} \cos x \, dx = dt$

$\Rightarrow \quad e^{\sqrt{\sin x}} \sqrt{\cot x}\sqrt{\cos x} \, dx = 2\,dt$

Hence $\int 2\,dt = 2t + C = 2e^{\sqrt{\sin x}} + C$

11. (b) $I = \int \frac{dx}{x\sqrt{x^6+1}}$. Put $x^6 + 1 = v^2$

$\Rightarrow \quad 6x^5\,dx = 2v\,dv$

$\therefore \quad I = \frac{1}{3}\int \frac{v}{(v^2-1)v} dv$

$= \frac{1}{3}\int \frac{dv}{v^2-1} = \frac{1}{6}\log\left(\frac{\sqrt{x^6+1}-1}{\sqrt{x^6+1}+1}\right) + C$

12. (a) $\int \frac{dx}{(1+\sqrt{x})^{2010}}$

$= \int \frac{\sqrt{x}}{\sqrt{x}(1+\sqrt{x})^{2010}} dx$

$= 2\int \frac{t-1}{t^{2010}} dt \, (t = 1 + \sqrt{x})$

$= 2\left[\frac{1}{2009t^{2009}} - \frac{1}{2008t^{2008}}\right] + c$

$\Rightarrow \quad \alpha = 2009, \beta = 2008$

13. (a) $\frac{1}{\sqrt{2}}\int \frac{d(\sin x + \cos x)}{(\sin x + \cos x)^2 + 1} = \frac{1}{\sqrt{2}} \tan^{-1}(\sin x + \cos x) + B$

DPP 7.3

Single Correct Answer Type

1. (b) Put $x + \sqrt{x^2+2} = t$

$\Rightarrow \quad x^2 + 2 = t^2 + x^2 - 2tx$

$\Rightarrow \quad x = \frac{1}{2}\left(t - \frac{2}{t}\right)$

So $\quad I = \int \sqrt{x + \sqrt{x^2+2}}\, dx$

$= \frac{1}{2}\int t^{1/2}\left(1 + \frac{2}{t^2}\right) dt$

$= \frac{1}{2}\int t^{1/2} dt + \int t^{-3/2} dt = \frac{1}{3}t^{3/2} - \frac{2}{\sqrt{t}} + c$

2. (a) $I = \int \frac{x^2+1}{x\sqrt{x^2+2x-1}\sqrt{1-x^2-x}} dx$

$= \int \frac{1 + \frac{1}{x^2}}{\sqrt{x+2-\frac{1}{x}}\sqrt{\frac{1}{x}-x-1}} dx$

Let $t = \sqrt{x+2-\frac{1}{x}}$

$\therefore \quad dt = \frac{1}{2\sqrt{x+2-\frac{1}{x}}}\left(1 + \frac{1}{x^2}\right) dx$

$\therefore \quad I = \int \frac{2dt}{\sqrt{1-t^2}}$

$= 2\sin^{-1} t + c$

$= 2\sin^{-1}\sqrt{x - \frac{1}{x} + 2} + c$

3. (c) $I = \int \frac{1}{x^2\sqrt{16-x^2}} dx$

Put $x = \frac{1}{t}$;

$dx = -\frac{1}{t^2} dt \therefore I = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t} \times \frac{1}{t^2}\sqrt{16t^2-1}} = \int \frac{-t\,dt}{\sqrt{16t^2-1}}$

Let $16t^2 - 1 = u^2$; $32t\,dt = 2u\,du$;

$t\,dt = \frac{u}{16} du \therefore I = -\frac{1}{16}\int \frac{u\,du}{u} = -\frac{u}{16} + C = -\frac{\sqrt{16-x^2}}{16x} + C$

4. (c) $I = \int \frac{3(\tan x - 1)\sec^2 x}{(\tan x + 1)\sqrt{\tan^3 x + \tan^2 x + \tan x}} dx$

Put $\tan x = t$

$\Rightarrow \quad I = 3\int \frac{1 - \frac{1}{t^2}}{\left(t + 2 + \frac{1}{t}\right)\sqrt{t + \frac{1}{t} + 1}} dt$

Put $\quad t + \frac{1}{t} + 1 = z^2$

$\Rightarrow \quad I = 3.2 \tan^{-1}(\sqrt{\cot x + \tan x + 1}) + C$

$\Rightarrow \quad K = 6$

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5. (a) Let $x = \sin\theta$; $dx = \cos\theta d\theta$

$$\therefore \quad I = \int \frac{(\cos\theta - \sin\theta)\cos\theta\, d\theta}{\cos\theta(1+\sin\theta\cos\theta)}$$

$$= 2\int \frac{(\cos\theta - \sin\theta)\, d\theta}{1+(\cos\theta+\sin\theta)^2}$$

$$= 2\tan^{-1}(\cos\theta+\sin\theta)+c$$

$$= 2\tan^{-1}(x+\sqrt{1-x^2})+c$$

6. (b) $I = \int \frac{3x^2+2x}{(x^3+x^2+1)^2+4}dx$

Put $x^3+x^2+1 = t$

$$\therefore \quad I = \int \frac{1}{t^2+4}dt$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x^3+x^2+1}{2}\right)+c$$

7. (c) Let $\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}dx$

Putting $x = \sin^2 t$ and $dx = 2\sin t\cos t\, dt$, we get

$$I = \int \frac{2\sin t\cos t dt}{(1+\sin t)\sqrt{\sin^2 t - \sin^4 t}}$$

$$\Rightarrow \quad I = 2\int \frac{1}{1+\sin t}dt = 2\int \frac{1-\sin t}{\cos^2 t}dt$$

$$= 2\int(\sec^2 t - \sec t\tan t)dt$$

$$= 2(\tan t - \sec t)+c$$

$$\Rightarrow \quad I = 2\left(\sqrt{\frac{x}{1-x}} - \frac{1}{\sqrt{1-x}}\right)+c$$

8. (b) $f(x) = \int \frac{5x^8+7x^6}{x^{14}\left(2+\frac{1}{x^7}+\frac{1}{x^5}\right)^2}dx$

$$= \int \frac{\frac{5}{x^6}+\frac{7}{x^8}}{\left(2+\frac{1}{x^7}+\frac{1}{x^5}\right)^2}dx$$

Put, $\quad 2+\frac{1}{x^7}+\frac{1}{x^5} = t$

$$\Rightarrow \quad f(x) = -\int \frac{dt}{t^2}$$

$$= \frac{1}{t}+c$$

$$= \left(\frac{x^7}{2x^7+x^2+1}\right)+c$$

$f(0) = 0$

$\Rightarrow \quad c = 0$

$$\Rightarrow \quad f(x) = \left(\frac{x^7}{2x^7+x^2+1}\right)$$

$$\Rightarrow \quad f(1) = \frac{1}{4}$$

9. (c) $x^4\sqrt{a^2+x^2} = x^5\sqrt{\left(\frac{a}{x}\right)^2+1}$

$$\therefore \quad I = \int \frac{\frac{dx}{x^3}}{x^2\sqrt{\left(\frac{a}{x}\right)^2+1}}$$

Let $\frac{a^2}{x^2} = t$, so $\frac{-2a^2}{x^3}dx = dt$

$$\therefore \quad I = -\frac{1}{2a^4}\int \frac{t}{\sqrt{t+1}}dt = -\frac{1}{2a^4}\int \frac{t+1-1}{\sqrt{t+1}}dt$$

$$= \frac{1}{a^4}\left[\sqrt{t+1} - \frac{1}{3}(t+1)^{3/2}\right]+c$$

$$= \frac{1}{a^4}\left[\frac{\sqrt{a^2+x^2}}{x} - \frac{1}{3x^3}(a^2+x^2)^{3/2}\right]+c$$

10. (b) $I = \int x^{24}(1+x+x^2)^6\left\{x^3(4+5x+6x^2)\right\}dx$

$$= \int (x^4+x^5+x^6)^6.(6x^5+5x^4+4x^3)dx$$

$$= \frac{x^{28}(1+x+x^2)^7}{7}+C$$

11. (b) $I = \int \frac{x^2(1-\log x)}{(\log x)^4 - x^4}dx$

$$= \int \frac{1-\log x}{x^2\left(\left(\frac{\log x}{x}\right)^4 - 1\right)}dx$$

Put $\frac{\log x}{x} = t \Rightarrow \frac{1-\log x}{x^2} = dt$

$$\therefore \quad I = \int \frac{dt}{(t^4-1)} = \int \frac{dt}{(t^2+1)(t^2-1)}$$

$$= \frac{1}{2}\int \frac{(t^2+1)-(t^2-1)}{(t^2+1)(t^2-1)}dt$$

$$= \frac{1}{2}\left(\int \frac{dt}{t^2-1} - \int \frac{dt}{t^2+1}\right) = \frac{1}{2}\left(\frac{1}{2}\ln\frac{t-1}{t+1} - \tan^{-1}t\right)$$

$$= \frac{1}{4}\ln\left(\frac{\ln x - x}{\ln x + x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$$

12. (b) We have $\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}}dx$

$$= \int \frac{x\left(x^2-1\right)}{(x^2+1)(x+1)^2\sqrt{x^3+x^2+x}}dx$$

$$= \int \frac{x^3\left(1-\frac{1}{x^2}\right)}{x^3\left(x+\frac{1}{x}\right)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\sqrt{x+\frac{1}{x}+1}}dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}dx$$

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$$I=\int\frac{2t}{(t^2-1)(t^2+1)\sqrt{t^2}}dt$$

where $x+\frac{1}{x}+1=t^2$

$$=\int\frac{2}{(t^2-1)(t^2+1)}dt$$

$$=\int\frac{1}{t^2-1}dt-\int\frac{1}{t^2+1}dt$$

$$=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|-\tan^{-1}t+c$$

$$=\frac{1}{2}\log\left|\frac{\sqrt{x+\frac{1}{x}+1}-1}{\sqrt{x+\frac{1}{x}+1}+1}\right|-\tan^{-1}\sqrt{x+\frac{1}{x}+1}+c$$

DPP 7.4

Subjective Type

1. $\int\frac{(x-1)^2}{x^4+x^2+1}dx$

$$=\int\frac{x^2+1}{x^4+x^2+1}dx-\int\frac{2x}{x^4+x^2+1}dx$$

$$=\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3}dx-\int\frac{2x}{\left(x^2+\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{x-1/x}{\sqrt{3}}-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$$

2. $I=\int\frac{x^2-4}{x^4+9x^2+16}dx$

Divide both numerator and denominator by x^2

$$I=\int\frac{\left(1-\frac{4}{x^2}\right)}{x^2+9+\frac{16}{x^2}}dx=\int\frac{\left(1-\frac{4}{x^2}\right)}{\left(x+\frac{4}{x}\right)^2+1}dx$$

Put $u=x+\frac{4}{x}$, $du=\left(1-\frac{4}{x^2}\right)dx$

$$I=\int\frac{du}{u^2+1}=\tan^{-1}(u)+B$$

$$=\tan^{-1}\left(x+\frac{4}{x}\right)+C$$

Single Correct Answer Type

3. (a) $\int\frac{dx^3}{x^3(x^n+1)}$

$$=3\int\frac{x^2dx}{x^3(x^n+1)}$$

$$=3\int\frac{dx}{x(x^n+1)}$$

$$=\int\frac{x^{n-1}}{x^n(x^n+1)}dx$$

$$=\frac{3}{n}\int\frac{dt}{t(t+1)}$$

$$=\frac{3}{n}\int\frac{(t+1)-t}{t(t+1)}dt$$

$$=\frac{3}{n}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$=\frac{3}{n}(\ln t-\ln(t+1))+c$$

$$=\frac{3}{n}\ln\left(\frac{t}{t+1}\right)+C$$

4. (b) $\int\frac{(x+1)^2}{x(x^2+1)}dx=\int\frac{x^2+1+2x}{x(x^2+1)}dx$

$$=\int\frac{x^2+1}{x(x^2+1)}dx+2\int\frac{x}{x(x^2+1)}dx$$

$$=\int\frac{dx}{x}+2\int\frac{dx}{x^2+1}=\log_e x+2\tan^{-1}x+c$$

5. (a) $I=\frac{1}{2}\int\frac{t-1}{1+t^3}dt$ (where $t=x^2$)

$$=\frac{1}{2}\int\left(\frac{-2}{3(1+t)}+\frac{1}{3}\frac{(2t-1)}{t^2-t+1}\right)dt$$ (After partial fractions)

$$=\frac{1}{6}\cdot\ln\frac{x^4-x^2+1}{(x^2+1)^2}+C$$

6. (c) $\int\frac{x^3-1}{(x^4+1)(x+1)}dx=\int\frac{(x^4+x^3)-(x^4+1)}{(x^4+1)(x+1)}dx$

$$=\int\frac{x^3}{x^4+1}dx-\int\frac{1}{x+1}dx$$

$$=\frac{1}{4}\ln(x^4+1)-\ln(x+1)+c$$

7. (b) $I=\int\frac{\cos^3x}{\sin^2x+\sin x}dx$

$$=\int\frac{\cos x\cdot(1-\sin^2x)}{\sin x(1+\sin x)}dx$$

Put $\sin x=t$, then $\cos x\,dx=dt$

$\Rightarrow$ $$I=\int\frac{(1-t)(1+t)dt}{t(1+t)}$$

$$=\int\frac{(1-t)dt}{t}$$

$$=\int\left(\frac{1}{t}-1\right)dt$$

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$= \log_e|t| - t + C$

$= \log_e|\sin x| - \sin x + C$

8. (b) $\int \frac{x^3-1}{x^3+x}dx = \int\left(1 - \frac{1}{1+x^2} - \frac{1}{x} + \frac{x}{x^2+1}\right)dx$

$= x - \tan^{-1}x - \log_e|x| + \frac{1}{2}\log_e(x^2+1) + C$

Comprehension Type

9. (d) $\int(f(x) - 2g(x))dx$

$= \int \frac{e^x(e^{2x}-2)}{e^{4x}+8e^{2x}+4}dx$ put $e^x = t$

$= \int \frac{1-\frac{2}{t^2}}{\left(t+\frac{2}{t}\right)^2+4}dt = \frac{1}{2}\tan^{-1}\left(\frac{t+\frac{2}{t}}{2}\right)+C$

10. (b) $f(x) + 2g(x) = \int \frac{e^x(e^{2x}+2)}{e^{4x}+8e^{2x}+4}dx$ put $e^x = t$

$= \int \frac{1+\frac{2}{t^2}}{\left(t-\frac{2}{t}\right)^2+12}dt = \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{2}{t}}{2\sqrt{3}}\right)+C$

DPP 7.5

Subjective Type

1. $\int \cos 2x \log_e(1+\tan x)\,dx$

$= \frac{1}{2}\sin 2x \log_e(1+\tan x) - \frac{1}{2}\int \frac{\sec^2 x}{(1+\tan x)}\sin 2x\,dx$

$= \frac{1}{2}\sin 2x \log_e(1+\tan x) - \int \frac{\tan x + 1 - 1}{1+\tan x}dx$

$= \frac{1}{2}\sin 2x \log_e(1+\tan x) - x + \frac{1}{2}\int \frac{2\cos x + \sin x - \sin x}{\sin x + \cos x}dx$

$= \frac{1}{2}\sin 2x \log_e(1+\tan x) - x + \frac{x}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}dx$

$= \frac{1}{2}[\sin 2x \log_e(1+\tan x) - x + \log|\sin x + \cos x|] + C$

2. $\int \sin 4x \cdot e^{\tan^2 x}dx = 4\int \sin x \cos x \cos 2x\, e^{\tan^2 x}dx$

$= 4\int \tan x \cdot \sec^2 x \cdot \cos^4 x \cdot \cos 2x\, e^{\tan^2 x}dx$

$= 2\int \frac{1}{(1+t)^2}\frac{1-t}{1+t}e^t\,dt$

(Putting, $\tan^2 x = t \Rightarrow 2\tan x \cdot \sec^2 x\,dx = dt$)

$= -2\int \frac{(t+1)-2}{(1+t)^3}e^t\,dt$

$= -2\int e^t\left(\frac{1}{(1+t)^2} + \frac{-2}{(1+t)^3}\right)dt$

$= \frac{-2e^t}{(1+t)^2} + C$

$= -2e^{\tan^2 x}\cos^4 x + C$

3. $\int \frac{\log_e(1+\sin^2 x)dx}{\cos^2 x}$

$= \log_e(1+\sin^2 x)\cdot\tan x - 2\int \frac{\sin^2 x + 1 - 1}{1+\sin^2 x}dx$

$= \log_e(1+\sin^2 x)\cdot\tan x - 2x + 2\int \frac{\sec^2 x\,dx}{1+2\tan^2 x}$

$= \log_e(1+\sin^2 x)\cdot\tan x - 2x + \sqrt{2}\tan^{-1}(\sqrt{2}\tan x) + C$

4. $\int \frac{x\log_e x}{(x^2-1)^{3/2}}dx$

$= \log_e x\int \frac{x}{(x^2-1)^{3/2}}dx - \int\left(\frac{1}{x}\left(\int \frac{x}{(x^2-1)^{3/2}}dx\right)\right)dx$

$= \frac{-\log_e x}{\sqrt{x^2-1}} + \int \frac{1}{x\sqrt{x^2-1}}dx$

$= \frac{-\log_e x}{\sqrt{x^2-1}} + \sec^{-1}x + C$

Single Correct Answer Type

5. (a) $I = \int \frac{(2\cos x + 1)}{(2+\cos x)^2}dx$

$= \int \frac{(2+\cos x)\cos x + \sin^2 x}{(2+\cos x)^2}dx$

$= \int \frac{\cos x}{2+\cos x}dx - \int \frac{-\sin^2 x}{(2+\cos x)^2}dx$

$= \frac{\sin x}{2+\cos x} + C$

6. (c) $I = \int f^{-1}(x)dx = f^{-1}(x)\cdot x - \int x\cdot((f^{-1})(x))'\,dx$

Put $f^{-1}(x) = t$

$\therefore \quad I = f^{-1}(x)\cdot x - \int f(t)\cdot dt$

$\therefore \quad I = f^{-1}(x)\cdot x - g(t) + C$

$= x\cdot f^{-1}(x) - g(f^{-1}(x)) + C$

7. (a) $t = \cot x \Rightarrow dt = -\text{cosec}^2 x\,dx$

$I = \int \frac{e^{\cot x}}{\sin^2 x}\left(\ln \text{cosec}^2 x + \frac{2\tan x}{1+\tan^2 x}\right)dx$

$= -\int e^t\left(\ln(1+t^2) + \frac{\frac{2}{t}}{1+\frac{1}{t^2}}\right)dt$

$= -e^t\ln|1+t^2| + c = -2e^{\cot x}\ln|\text{cosec } x| + c$

$= 2e^{\cot x}\ln|\sin x| + c$

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9. (c) $\int \frac{1-7\cos^2 x}{\sin^7 x \cos^2 x}dx = \int\left(\frac{\sec^2 x}{\sin^7 x} - \frac{7}{\sin^7 x}\right)dx$

$= \int \frac{\sec^2 x}{\sin^7 x}dx - \int \frac{7}{\sin^7}dx = I_1 + I_2$

$I_1 = \int \frac{\sec^2 x}{\sin^7 x}dx$

$= \frac{\tan x}{\sin^7 x} + 7\int \frac{\tan x \cos x}{\sin^8 x}dx$

$= \frac{\tan x}{\sin^7 x} - I_2$

$\therefore \quad I_1 + I_2 = \frac{\tan x}{\sin^7 x} + C$

$\Rightarrow \quad f(x) = \tan x$

10. (a) If $\int e^x\left(\tan^{-1} x + \frac{2x}{(1+x^2)^2}\right)dx$

$= \int e^x\left(\tan^{-1} x - \frac{1}{1+x^2} + \frac{1}{1+x^2} + \frac{2x}{(1+x^2)^2}\right)dx$

$= e^x\left(\tan^{-1} x - \frac{1}{1+x^2}\right) + c$

$f(0) = 0 \Rightarrow c = 1$

$\Rightarrow \quad f(1) = e\left(\frac{\pi}{4} - \frac{1}{2}\right) + 1$

10. (d) $\int \frac{e^x(x-1)(x-\ln x)}{x^2}dx$

$= \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)\left(\ln \frac{e^x}{x}\right)dx$

$= \int \ln \frac{e^x}{x} d\left(\frac{e^x}{x}\right)$

$= \ln\left(\frac{e^x}{x}\right)\frac{e^x}{x} - \frac{e^x}{x} + c$

$= \frac{e^x}{x}(x - \ln x - 1) + c$

11. (a) $I = \int(\sin(100x + x) \cdot (\sin x)^{99}\,dx$

$= \int[(\sin(100x)\cos x + \cos 100x \cdot \sin x)(\sin x)^{99}]dx$

$= \int \underbrace{\sin(100x)}_{\text{I}} \underbrace{\cos x \cdot (\sin x)^{99}}_{\text{II}}\,dx + \int \cos(100x) \cdot (\sin x)^{100}\,dx$

$= \frac{\sin(100x)(\sin x)^{100}}{100} - \frac{100}{100}\int \cos(100x)(\sin x)^{100}\,dx$

$+ \int \cos(100x)(\sin x)^{100}\,dx$

$= \frac{\sin(100x)(\sin x)^{100}}{100} + C$

Multiple Correct Answers Type

12. (b, c)

$I = \int\{(e^{2x} - e^{-2x})\ln(e^x + e^{-x}) - (e^{2x} - e^{-2x})\ln(e^x - e^{-x})\}dx$

$= \int t \ln t\,dt - \int u \ln u\,du$ (where $t = e^x + e^{-x}$ and $u = e^x - e^{-x}$)

$= \frac{t^2}{2}\ln t - \frac{t^2}{4} - \frac{u^2}{2}\ln u + \frac{u^2}{4} + C$

13. (b, d)

$\int \frac{xe^x}{\sqrt{1+e^x}}dx$

$= x(2\sqrt{1+e^x}) - 2\int \sqrt{1+e^x}\,dx$

$= 2x\sqrt{1+e^x} - 2\int \frac{2t^2}{t^2-1}dt$ (Putting $t = \sqrt{1+e^x}$)

$= 2x\sqrt{1+e^x} - 4\left(t + \frac{1}{2}\ln\frac{t-1}{t+1}\right) + C$

$= 2x\sqrt{1+e^x} - 4(\sqrt{1+e^x}) - 2\ln\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + C$

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CHAPTER 8

DPP 8.1

Single Correct Answer Type

1. (a) $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots+\frac{1}{8n}\right]$

$=\lim\limits_{n\to\infty}\left[\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots+\frac{n^2}{(n+n)^3}\right]$

$=\lim\limits_{n\to\infty}\sum\limits_{r=0}^{n}\frac{n^2}{(n+r)^3}$

$=\lim\limits_{n\to\infty}\sum\limits_{r=0}^{n}\frac{1}{n}\frac{1}{\left(1+\frac{r}{n}\right)^3}$

$=\int\limits_0^1\frac{dx}{(1+x^3)}=\left[-\frac{1}{2(1+x)^2}\right]_0^1$

$=-\frac{1}{2}\left(\frac{1}{4}-1\right)=\frac{3}{8}$

2. (c) $S=\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{\pi}{n}\cdot\frac{1}{\sin\left\{\frac{\pi}{4}+\frac{\pi r}{4n}\right\}}$

$=\pi\int\limits_0^1\frac{dx}{\sin\left(\frac{\pi}{4}x+\frac{\pi}{4}\right)}=4\log_e(\sqrt{2}-1)$

3. (a) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{k^{1/a\left\{n^{\frac{a-1}{a}}+k^{\frac{a-1}{a}}\right\}}}{n^{a+1}}$

$=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n}\cdot\left\{\left(\frac{k}{n}\right)^{1/a}+\left(\frac{k}{n}\right)^{a}\right\}$

$=\int\limits_0^1(x^{1/a}+x^a)\,dx$

$=\left\{\frac{x^{(1/a)+1}}{\frac{1}{a}+1}+\frac{x^{a+1}}{a+1}\right\}_0^1$

$=\frac{a}{a+1}+\frac{1}{a+1}=1$

4. (c) $\int\limits_0^3(3ax^2+2bx+c)\,dx=\int\limits_1^3(3ax^2+2bx+c)\,dx$

$\Rightarrow\quad\int\limits_0^1(3ax^2+2bx+c)\,dx+\int\limits_1^3(3ax^2+2bx+c)\,dx$

$=\int\limits_1^3(3ax^2+2bx+c)$

$\Rightarrow\quad\int\limits_0^1(3ax^2+2bx+c)dx=0$

$\Rightarrow\quad\left[\frac{3ax^3}{3}+\frac{2bx^2}{2}+cx\right]_0^1=0$

$\Rightarrow\quad a+b+c=0$

5. (a) $\int\limits_{-2}^{x}|\cos x|dx=\int\limits_{-2}^{-\pi/2}|\cos x|\,dx+\int\limits_{-\pi/2}^{x}|\cos x|dx$

$=\int\limits_{-2}^{-\pi/2}-\cos x\,dx+\int\limits_{-\pi/2}^{x}\cos x\,dx$

$=1-\sin 2+\sin x+1$

$\therefore\quad\int\limits_{-2}^{x}|\cos x|\,dx=0$

$\Rightarrow\quad 1-\sin 2+\sin x+1=0$

$\Rightarrow\quad \sin x=\sin 2-2<-1$

$\therefore\quad$ No solution in $\left(0,\frac{\pi}{2}\right)$.

6. (b)

$I=\int\limits_0^1 e^{2x}e^{e^x}dx$

Let $e^x=1$

$\Rightarrow\quad I=\int\limits_0^e te^t\,dt=(te^t-e^t)_1^e=e^e(e-1)$

7. (b) $I=\int\limits_0^{\tan 1}f(x)dx+\int\limits_{\tan 1}^{\infty}f(x)dx$

$=\int\limits_0^{\tan 1}\cos x\,dx+0$

$=\sin(\tan 1)$

8. (a)

$I=\int\limits_0^1\frac{dx}{(x+1)\sqrt{(x+1)(3(x+1)-2)}}$

Put $x+1=\frac{1}{t}$

$\therefore\quad I=\int\limits_{1/2}^{1}\frac{dt}{\sqrt{3-2t}}=\sqrt{2}-1$

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9. (a) Let $I = \int_0^3 x \cdot f(x^2)\,dx$

Put $x^2 = t$

$\Rightarrow \quad 2x \cdot dx = dt$

$\Rightarrow \quad I = \frac{1}{2}\int_0^9 f(t)\,dt = \frac{1}{2}\cdot 4 = 2$

10. (b) $\lim_{t\to 0}\int_0^{2\pi}\left|\frac{\sin(x+t)-\sin x}{t}\right|dx$

$$= \int_0^{2\pi}\left(\lim_{t\to 0}\left|\frac{2\cos\left(x+\frac{t}{2}\right)\sin\frac{t}{2}}{t}\right|\right)dt$$

$$= \int_0^{2\pi}|\cos x|\,dx = 4$$

DPP 8.2

Single Correct Answer Type

1. (b) $I = \int_{-1}^{1}(1+x)^{1/2}(1-x)^{3/2}\,dx$

$\Rightarrow \quad I = \int_{-1}^{1}(1-x)^{1/2}(1+x)^{3/2}\,dx$

$\Rightarrow \quad 2I = \int_{-1}^{1}(1+x)^{1/2}(1-x)^{1/2}[(1-x)+(1+x)]dx$

$\Rightarrow \quad 2I = 2\int_{-1}^{1}\sqrt{1-x^2}\,dx$

$\Rightarrow \quad I = 2\int_0^1\sqrt{1-x^2}\,dx$

$$= \frac{\pi}{2}$$

2. (b) $I = \int_0^{\pi} x(\sin^2(\sin x) + \cos^2(\cos x))dx$ (1)

$\therefore \quad I = \int_0^{\pi}(\pi - x)(\sin^2(\sin x) + \cos^2(\cos x))\,dx$ (2)

Adding (1) and (2)

$\Rightarrow \quad 2I = 2\pi\int_0^{\frac{\pi}{2}}(\sin^2(\sin x) + \cos^2(\cos x))\,dx$

$\Rightarrow \quad I = \pi\int_0^{\frac{\pi}{2}}(\sin^2(\sin x) + \cos^2(\cos x))\,dx$ (3)

$\Rightarrow \quad I = \pi\int_0^{\frac{\pi}{2}}(\sin^2(\cos x) + \cos^2(\sin x))\,dx$ (4)

Adding (3) and (4)

$\Rightarrow \quad 2I = \pi\int_0^{\frac{\pi}{2}}2\,dx \Rightarrow I = \frac{\pi^2}{2}$

3. (b) $I = \int_0^a \ln(\cot a + \tan x)\,dx$

$$= \int_0^a \ln\left(\frac{\cos(a-x)}{\sin a\cos x}\right)dx \quad (1)$$

$\therefore \quad I = \int_0^a \ln\left(\frac{\cos x}{\sin a\cos(a-x)}\right)dx$ (2)

Adding (1) and (2) we get $2I = \int_0^a \ln\left(\frac{1}{\sin^2 a}\right)dx$

$$= -2\int_0^a \ln(\sin a)\,dx$$

$$= -2a\ln(\sin a)$$

4. (d) Put $y = 0 \Rightarrow f(x+1) = f(x)$

Thus $f(x)$ is periodic with period '1'

$\therefore \quad \int_0^{10} f(10-x)\,dx$

$$= \int_0^{10} f(10-(10-x))\,dx$$

$$= \int_0^{10} f(x)\,dx$$

$$= 10\int_0^1 f(x)\,dx$$

5. (a) $u = \int_0^{\pi/2}\cos\left(\frac{2\pi}{3}\sin^2 x\right)dx$

$u = \int_0^{\pi/2}\cos\left(\frac{2\pi}{3}\cos^2 x\right)dx$

$\Rightarrow \quad 2u = \int_0^{\pi/2}\left[\cos\left(\frac{2\pi}{3}\sin^2 x\right) + \cos\left(\frac{2\pi}{3}\cdot\cos^2 x\right)\right]dx$

$\Rightarrow \quad 2u = \int_0^{\pi/2} 2\cos\frac{\pi}{3}\cdot\cos\left(\frac{\pi}{3}\cos 2x\right)dx$

$\Rightarrow \quad u = \frac{1}{2}\int_0^{\pi}\cos\left(\frac{\pi}{3}\cos t\right)dt$ [Put $2x = t$]

$$= \int_0^{\pi/2}\cos\left(\frac{\pi}{3}\cos t\right)dt$$

6. (a) $I = \int_0^{100\pi}(\tan x + \tan 2x + \tan 3x + \ldots + \tan 10x)\,dx$

Period of $f(x) = \tan x + \tan 2x + \ldots + \tan 10x$ is π

$\therefore \quad I = 100\int_0^{\pi}(\tan x + \tan 2x + \tan 3x + \ldots + \tan 10x)\,dx$

Now, $f(x) = -f(\pi - x)$

$\therefore \quad I = 0$

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7. (c)

$$I=\int_0^{\pi/2}\sin x\sin 2x\sin 3x\sin 4x\,dx=\int_0^{\pi/2}\cos x\sin 2x\cos 3x\sin 4x\,dx$$

$$\Rightarrow \quad 2I=\int_0^{\pi/2}\cos 2x\sin 2x\sin 4x\,dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\sin^2 4x\,dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{1-\cos 8x}{2}dx$$

$$=\frac{1}{4}\left[x-\frac{\sin 8x}{8}\right]_0^{\pi/2}=\frac{\pi}{8}$$

8. (d) $I=\int_1^{2013}(x-1)(x-2)(x-3)\ldots(x-2013)dx$

Using $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

$$\int_1^{2013}(2013-x)(2012-x)\ldots(1-x)=-I$$

$$\Rightarrow \quad 2I=0\Rightarrow I=0$$

9. (c) $\int_1^4 f'(x)\,dx=[xf'(x)]_1^4-\int_1^4 xf''(x)$

Now $I=\int_1^4 x\,f''(x)\,dx=\int_1^4(5-x)f''(5-x)\,dx$

$$=5\int_1^4 f''(x)\,dx-I$$

$$\therefore \quad I=\frac{5}{2}[f'(4)-f'(1)]$$

$$\therefore \quad \int_1^4 f'(x)\,dx=\frac{3}{2}[f'(4)+f'(1)]$$

Now, $f''(x)=f''(5-x)$

$\Rightarrow$ $f'(x)=-f'(5-x)+c$

$\Rightarrow$ $f'(0)+f'(5)=c\Rightarrow c=8$

so $f'(x)+f'(5-x)=8\Rightarrow f'(4)+f'(1)=8$

10. (a) Let $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{e^x\sec^2 x\,dx}{e^{2x}-1}$

If $f(x)=\frac{e^x\sec^2 x}{e^{2x}-1}$

$$\therefore \quad f(-x)=\frac{e^{-x}\sec^2 x}{e^{-2x}-1}$$

$$=\frac{e^x\sec^2 x}{1-e^{2x}}$$

$$=-\frac{e^x\sec^2 x}{e^{2x}-1}$$

$$=-f(x)$$

$\therefore f(x)$ is an odd function.

$\therefore \quad I=0$

11. (b)

$$I=\int_{-a}^{a}\frac{f(x)}{b^{g(x)}+1}dx \qquad \text{(i)}$$

$$=\int_{-a}^{a}\frac{f(-x)}{b^{g(-x)}+1}dx$$

$$=\int_{-a}^{a}\frac{f(x)}{b^{-g(x)}+1}dx$$

$$=\int_{-a}^{a}\frac{f(x)\cdot b^{g(x)}}{1+b^{g(x)}}dx$$

$$\therefore \quad I=\int_{-a}^{a}\frac{f(x)\cdot b^{g(x)}}{1+b^{g(x)}}dx \qquad \text{(ii)}$$

Adding (i) and (ii), $2I=\int_{-a}^{a}\frac{f(x)}{1+b^{g(x)}}dx+\int_{-a}^{a}\frac{f(x)\cdot b^{g(x)}}{1+b^{g(x)}}dx$

$$=\int_{-a}^{a}f(x)dx$$

$$\therefore \quad I=\int_0^a f(x)\,dx,$$

which is independent of $g(x)$.

12. (b)

$$\int_0^{4\pi}\ln|13\sin x+3\sqrt{3}\cos x|\,dx$$

$$=4\int_0^{\pi}\ln\left|14\sin\left(x+\tan^{-1}\frac{3\sqrt{3}}{13}\right)\right|dx$$

$$=8\int_0^{\pi/2}(\ln 14+\log|\sin x|)\,dx$$

$$=4\pi(\ln 14-\ln 2)=4\pi\ln 7.$$

13. (b) Given $f(x)+f(-x)=2$

Let $f(x)=t$

$\therefore \quad f(-x)=2-t$

$\therefore \quad f^{-1}(t)+f^{-1}(2-t)=0$

$$I=\int_{1-x}^{1+x}f^{-1}(t)\,dt=\int_{1-x}^{1+x}f^{-1}(2-t)\,dt$$

$$\therefore \quad 2I=\int_{1-x}^{1+x}(f^{-1}(t)+f^{-1}(2-t))\,dt$$

$$I=0$$

Multiple Correct Answers Type

14. (b, d)

$$A(x+y)=A(x)A(y)$$

$\Rightarrow$ $A(0+0)=A(0)A(0)$

$\Rightarrow$ $A(0)=1$

Put $y=-x$, we get

$$A(0)=A(x)A(-x) \qquad \text{(i)}$$

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$$B(-x)=\frac{A(-x)}{1+(A(-x))^2}$$
$$=\frac{\frac{1}{A(x)}}{1+\frac{1}{(A(x))^2}}$$
$$=\frac{A(x)}{1+(A(x))^2}$$
$$=B(x)$$

Thus, $B(x)$ is even.

$$\int_{-2010}^{2011} B(x)dx=\int_{-2010}^{2010} B(x)dx+\int_{2010}^{2011} B(x)dx$$
$$=2\int_{0}^{2010} B(x)dx+\int_{2010}^{2011} B(x)dx$$
$$=\int_{0}^{2010} B(x)dx+\int_{0}^{2011} B(x)dx$$

15. (b, c, d)

$$f(x)=\int_0^{\pi}\cos t\cos(x-t)\,dt \qquad \text{(i)}$$
$$=\int_0^{\pi}-\cos t\cos(x-\pi+t)\,dt$$

$$\therefore \quad f(x)=\int_0^{\pi}\cos t\cdot\cos(x+t)\,dt \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2f(x)=\int_0^{\pi}\cos t(2\cos x\cdot\cos t)\,dt$$

$$\therefore \quad f(x)=\cos x\int_0^{\pi}\cos^2 t\,dt=2\cos x\int_0^{\pi/2}\cos^2 t\,dt$$

$\therefore$ $f(x)=\dfrac{\pi\cos x}{2}$, which is continuous and differentiable, having maximum value $\pi/2$ and minimum value $-\pi/2$. Also $f(x)$ satisfies all the conditions of Rolle's theorem in $[0, 2\pi]$.

DPP 8.3

Single Correct Answer Type

1. (c) $\displaystyle\int_{\log 2}^{x}\frac{du}{(e^u-1)^{1/2}}=\frac{\pi}{6}$

Put $e^u-1=t^2$

$$\Rightarrow \quad \int_1^{\sqrt{e^x-1}}\frac{2t}{1+t^2}dt=\frac{\pi}{6}$$

$$\Rightarrow \quad 2(\tan^{-1}t)_1^{\sqrt{e^x-1}}=\frac{\pi}{6}$$

$$\Rightarrow \quad \tan^{-1}\sqrt{e^x-1}-\frac{\pi}{4}=\frac{\pi}{12}$$

$$\Rightarrow \quad \sqrt{e^x-1}=\tan\frac{\pi}{3}\Rightarrow\sqrt{e^x-1}=\sqrt{3}\Rightarrow e^x=4.$$

2. (b) Let $\log_e x=t$

$$I=\int_1^{\log_e\pi^2}[\log_\pi e^t]\,dt$$

$$\Rightarrow \quad I=\int_1^{\log_e\pi^2}[t\log_\pi e]dt=\int_1^{\log_e\pi}0\,dt+\int_{\log_e\pi}^{2\log_e\pi}1dt=\log_e\pi$$

3. (d) Put $t^2=\sin\theta$ in numerator (Nr.)

$$\therefore \quad I_1=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\sin\theta}}d\theta$$

Put $t^2=\tan\alpha$ in denominator

$$\therefore \quad I_2=\int_0^1\frac{1}{\sqrt{1+t^4}}dt=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sqrt{2}}{\sqrt{\sin 2\alpha}}d\alpha=\frac{1}{\sqrt{2}}I_1$$

4. (a)

$$I=\int_a^b\frac{e^{x/a}-e^{b/x}}{x}dx$$

Put $\dfrac{x}{a}=\dfrac{b}{y}$

$$\Rightarrow \quad I=\int_b^a\frac{e^{\frac{b}{y}}-e^{\frac{y}{a}}}{\frac{ab}{y}}\left(-\frac{ab}{y^2}\right)dy$$
$$=\int_a^b\frac{b^{b/x}-e^{x/a}}{x}dx=-I$$

$$\Rightarrow \quad 2I=0$$
$$\Rightarrow \quad I=0$$

5. (b) $\displaystyle I=\int_0^{\infty}\frac{\log x}{a^2+x^2}dx$

Put $x=\dfrac{a^2}{y}, dx=-\dfrac{a^2}{y^2}dy$

$$\therefore \quad I=\int_{\infty}^{0}\frac{\log\left(\frac{a^2}{y}\right)}{a^2+\frac{a^4}{y^2}}\left(\frac{-a^2}{y^2}\right)dy$$
$$=\int_{\infty}^{0}\frac{(\log a^2-\log y)}{a^2+y^2}(-dy)$$
$$=\log(a^2)\int_0^{\infty}\frac{dy}{a^2+y^2}-\int_0^{\infty}\frac{\log y}{a^2+y^2}dy$$
$$=\log(a^2)\frac{1}{a}\left(\tan^{-1}\left(\frac{y}{a}\right)\right)_0^{\infty}-I$$

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$$\Rightarrow \quad 2I = \frac{2\log a}{a} \cdot \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi \log a}{2a}$$

6. (c) $I = \int_{1/3}^{3} \frac{1}{x} \log_e \left(\left| \frac{x + x^2 - 1}{x - x^2 + 1} \right| \right) dx$

Let $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$

$$\Rightarrow \quad I = -\int_{3}^{1/3} t \log_e \left(\left| \frac{\frac{1}{t} + \frac{1}{t^2} - 1}{\frac{1}{t} - \frac{1}{t^2} + 1} \right| \right) \frac{1}{t^2} dt$$

$$= \int_{1/3}^{3} \frac{1}{t} \log_e \left(\left| \frac{t - t^2 + 1}{t + t^2 - 1} \right| \right) dt$$

$$= -\int_{1/3}^{3} \frac{1}{x} \log_e \left(\left| \frac{x + x^2 - 1}{x - x^2 + 1} \right| \right) dx$$

$$\Rightarrow \quad I = -I \Rightarrow I = 0$$

7. (c) Let $x = t^2$

$$I_n = \int_0^1 2t(1 + t^2 + t^4 + \ldots + t^{2n-2})\,(1 + 3t^2 + 5t^4 + \ldots + (2n-3)\,t^{2n-4} + (2n-1)t^{2n-2})dt$$

$$= \int_0^1 2(t + t^3 + t^5 + \ldots + t^{2n-1})\,(1 + 3t^2 + 5t^4 + \ldots + (2n-3)\,t^{2n-4} + (2n-1)t^{2n-2})dt$$

Let $y = t + t^3 + t^5 + \ldots + t^{2n-1}$

$$\therefore \quad I_n = 2\int_0^n t\,dt = n^2$$

$$\therefore \quad \sqrt{I_9} = \sqrt{9^2} = 9$$

8. (b) $I = \int_1^c \frac{f(x)}{x} dx = \int_1^{\sqrt{c}} \frac{f(x)}{x} dx + \int_{\sqrt{c}}^{c} \frac{f(x)}{x} dx$

Put $x = \frac{c}{t}$ in second integral

$$\Rightarrow \quad dx = -\frac{c}{t^2} dt$$

$$\therefore \quad I = \int_1^{\sqrt{c}} \frac{f(x)}{x} dx + \int_{\sqrt{c}}^{1} \frac{f\left(\frac{c}{t}\right)}{t} (-dt)$$

$$= \int_1^{\sqrt{c}} \frac{f(x)}{x} dx + \int_1^{\sqrt{c}} \frac{f(x)}{x} dx = 6$$

9. (d) $I_1 = \int_0^\infty \frac{x^2\sqrt{x}}{(1+x)^6} dx$

Let $x = \frac{1}{t}$

$$\Rightarrow \quad I_1 = \int_\infty^0 \frac{\frac{1}{t^2}\sqrt{\frac{1}{t}}}{\left(1 + \frac{1}{t}\right)^6} \left(-\frac{1}{t^2} dt\right)$$

$$\Rightarrow \quad I_1 = \int_0^\infty \frac{t\sqrt{t}}{(1+t)^6} dt = I_2$$

Multiple Correct Answers Type

10. (a, b, c)

We have $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

For $\int_0^\infty e^{-2x^2} dx$, put $\sqrt{2}x = t$.

$$\therefore \quad \frac{1}{\sqrt{2}} \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2\sqrt{2}},$$

For $\int_0^\infty xe^{-x^2} dx$, put $x^2 = t$

$$\therefore \quad \int_0^\infty xe^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{-t} dt = \frac{1}{2}$$

For $\int_0^\infty x^2 e^{-x^2} dx = \int_0^\infty x(xe^{-x^2}) dx$, integrating by parts,

$$= \left\{ x\left(-\frac{1}{2}e^{-x^2}\right) \right\}_0^\infty + \frac{1}{2}\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$$

11. (b, c)

$e^{-a}[f(x + a) - f(1 + a)]$

$$= e^{-a}\left[\int_0^{x+a} \frac{e^t \cdot dt}{t} - \int_0^{1+a} \frac{e^t}{t} dt \right]$$

$$= e^{-a}\left[\int_0^{x+a} \frac{e^t \cdot dt}{t} + \int_{1+a}^{0} \frac{e^t}{t} dt \right]$$

$$= e^{-a}\left[\int_{1+a}^{x+a} \frac{e^t \cdot dt}{t} \right]$$

$$= e^{-a} \int_1^x \frac{e^{y+a}}{y+a} \cdot dy \qquad (\text{Put, } t = y + a,\ dt = dy)$$

$$= \int_1^x \frac{e^y \cdot dy}{y+a}$$

$$= \int_1^x \frac{e^t \cdot dt}{t+a}$$

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DPP 8.4

Single Correct Answer Type

1. (d) Differentiating both sides w.r.t. x, then

$$f(x^2(1+x)) \times (2x + 3x^2) = 1$$

At $x = 1 \Rightarrow f(2) = \frac{1}{5}$

2. (b) $f'(x) > 0 \Rightarrow \log_{0.5}\left(\frac{2x-8}{x-2}\right) > 0$

$$\Rightarrow \quad \frac{2x-8}{x-2} < 1$$

$$\Rightarrow \quad \frac{2x-8}{x-2} - 1 < 0$$

$$\Rightarrow \quad x \in (2, 6)$$

Also $\frac{2x-8}{x-2} > 0 \quad \Rightarrow \quad x < 0$ or $x > 4$

$$\Rightarrow \quad x \in (4, 6)$$

3. (a) $L = \lim_{t\to 0} \log_e \left(\frac{1}{t}\int_0^t (1 + a \sin bx)^{c/x}\, dx\right) \quad \left(\frac{0}{0} \text{ form}\right)$

$$= \log_e \left(\lim_{t\to 0} \frac{\int_0^t (1 + a \sin bx)^{c/x}\, dx}{t}\right) \quad \text{(Using L'Hopital Rule)}$$

$$= \log_e \left(\lim_{t\to 0} \frac{(1 + a \sin bt)^{c/t}}{1}\right)$$

$$= \log_e \left[\left(\lim_{t\to 0} (1 + a \sin bt)^{\frac{1}{a \sin bt}}\right)^{\frac{ac \sin bt}{t}}\right]$$

$$= \log_e e^{abc}$$

$$= abc$$

4. (a) $f(x) = \int_2^x \frac{dt}{1+t^4}$,

$$\Rightarrow \quad f'(x) = \frac{1}{1+x^4}$$

In [2, 3], apply mean value theorem to $f(x)$

$$\therefore \quad \frac{f(3) - f(2)}{3-2} = f'(c), \text{ where } c \in (2, 3)$$

$$\therefore \quad f(3) - 0 = \frac{1}{1+c^4}$$

Now $2 < c < 3$

$$17 < 1 + c^4 < 82$$

$$\Rightarrow \quad \frac{1}{17} > \frac{1}{1+c^4} > \frac{1}{82}$$

$$\Rightarrow \quad \frac{1}{82} < f(3) < \frac{1}{17}$$

5. (c)

$$\int_0^x f(x) \sin t\, dt = \text{constant}$$

Differentiate both side w.r.t. x

$$f'(x)(1 - \cos x) + f(x) \sin x = 0$$

$$\Rightarrow \quad \int \frac{f'(x)}{f(x)} dx = \int \frac{\sin x}{\cos x - 1} dx$$

$$\Rightarrow \quad \ln |f(x)| = -2 \ln \sin \frac{x}{2} + \ln c$$

$$\Rightarrow \quad f(x) = \frac{c}{\left(\sin \frac{x}{2}\right)^2}$$

$$f(\pi) = 2 \Rightarrow c = 2$$

$$\Rightarrow \quad f\left(\frac{\pi}{2}\right) = 4$$

6. (c) Since, I is independent of x, so $\frac{dI}{dx} = 0$

$$\Rightarrow \quad f(xy) \cdot y - f(x) = 0$$

$$\Rightarrow \quad f(x \cdot y) \cdot y = f(x)$$

Put, $y = \frac{2}{x}$; $f(2)\left(\frac{2}{x}\right) = f(x)$

$$\Rightarrow \quad f(x) = \frac{4}{x}$$

$$\Rightarrow \quad \int_1^e \frac{4}{x} \cdot dx = 4 \ln e = 4$$

7. (d) $f(a) = \int_{a-1}^{a+1} \frac{1}{1+x^4} dx$

$$f'(a) = \frac{1}{1+(a+1)^4} - \frac{1}{1+(a-1)^4} = 0 \text{ at } a = 0$$

$f''(0) < 0$ at $a = 0$

$\therefore$ f(a) has a local maximum at $a = 0$.

8. (d) $\lim_{x\to 0} \frac{\int_0^x \frac{t^2}{\sqrt{a+t}} dt}{x - \sin x}$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}\left(\int_0^x \frac{t^2}{\sqrt{a+t}} dt\right)}{1 - \cos x}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{\sqrt{a+x}}}{2 \sin^2 \frac{x}{2}} \quad \text{(by L' Hopital's Rule)}$$

$$= \lim_{x\to 0} \frac{1}{2\sqrt{a+x}} \cdot \left(\frac{\frac{x}{2}}{\sin \frac{x}{2}}\right)^2 \cdot 4$$

$$= \frac{1}{2\sqrt{a}} \cdot 1 \cdot 4 = 1 \Rightarrow a = 4.$$

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9. (a) $\int_0^x (f(t))^3\,dt = \frac{1}{x^2}\left(\int_0^x f(t)\,dt\right)^3$

$\therefore \quad \int_0^x (f(t))^3\,dt = \frac{1}{x^2}(g(x))^3$

Differentiating w.r.t. x,

$\therefore \quad (f(x))^3 = \frac{1}{x^2}3(g(x))^2 g'(x) - \frac{2}{x^3}(g(x))^3$

$\Rightarrow \quad \left(\frac{x\,g'(x)}{g(x)}\right)^3 - 3\left(\frac{x\cdot g'(x)}{g(x)}\right) + 2 = 0$

$\Rightarrow \quad \frac{xg'(x)}{g(x)} = 1 \text{ or } -2$

If $\frac{xg'(x)}{g(x)} = 1$

$\Rightarrow \quad xf(x) = \int_0^x f(t)\,dt$

$\Rightarrow \quad xf'(x) + f(x) = f(x)$

$\Rightarrow \quad f(x) = 1$

or $\quad xf'(x) + f(x) = -2f(x)$

$\frac{f'(x)}{f(x)} = \frac{-3}{x}$

$\log f(x) = -3 \log x + \log c$

$\Rightarrow \quad f(x) = c/x^3$

$\Rightarrow \quad f(1) = 1 \Rightarrow f(x) = 1/x^3$ (decreasing function)

10. (c) $g(x) = \int_0^x \left(t^2 \cdot \int_1^t f(u)\,du\right)dt$

$\Rightarrow \quad g'(x) = x^2\int_1^x f(u)\,du$ hence $g'(1) = 0$

$\Rightarrow \quad g''(x) = x^2 f(x) + \left(\int_1^x f(u)du\right)\cdot 2x$

$\Rightarrow \quad g''(1) = f(1) + 0 = 3$

$\therefore \quad g'(1) + g''(1) = 3$

Multiple Correct Answers Type

11. (a, c)

$f(2) = 0$

Also, $f'(x) = f(x^2 - 3x + 4)$

$f'(2) = f(2) = 0$

12. (a, b)

Put $x = 0 \Rightarrow e^0 - a\int_0^1 f(t)e^{-t}dt = 0$

$\Rightarrow \quad \int_0^1 f(t)e^{-t}dt = \frac{1}{a} \qquad \text{(i)}$

$\Rightarrow \quad \int_0^x f(t)\,dt = e^x - ae^{2x}\frac{1}{a} = e^x - e^{2x}$

Differentiating we get

$\Rightarrow \quad f(x) = e^x - 2e^{2x}$

From (i), we get $a = \frac{1}{3-2e}$.

13. (a, b, c)

$f(x) = \sin x + \int_0^x f'(t)\,(2\sin t - \sin^2 t)\,dt$

$\Rightarrow \quad f'(x) = \cos x + f'(x)\,(2\sin x - \sin^2 x)$

$\Rightarrow \quad (1 - 2\sin x + \sin^2 x)\,f'(x) = \cos x$

$\Rightarrow \quad f'(x) = \frac{\cos x}{(\sin x - 1)^2}$

$\Rightarrow \quad f(x) = \frac{1}{1-\sin x} + c$

since $f(0) = 0$ we have $c = -1$

$\Rightarrow \quad f(x) = \frac{\sin x}{1-\sin x}$

Comprehension Type

14. (b), 15. (c)

$f(x) = \int_0^x (5\ln(1+t^2) - 10t\tan^{-1}t + 16\sin t)\,dt.$

$\Rightarrow \quad f'(x) = 5\ln(1+x^2) - 10x\tan^{-1}x + 16\sin x$

$\Rightarrow \quad f''(x) = 2(8\cos x - 5\tan^{-1}x)$

$\Rightarrow \quad f'''(x) = -2\left(8\sin x + \frac{5}{1+x^2}\right) < 0\ \forall\, x \in (0, 1)$

So, $f''(x)$ is decreasing. $\forall\, x \in (0, 1)$

$\Rightarrow \quad f''(x) > f''(1) = 2\left(8\cos 1 - \frac{5\pi}{4}\right)$

$> 2\left(8\cos\frac{\pi}{3} - \frac{5\pi}{4}\right)$

$= 2\left(4 - \frac{5\pi}{4}\right) > 0$

So, $f'(x)$ is increasing, for $x > 0$, $f'(x) > f'(0) = 0$

So, $f(x)$ is increasing, for $x > 0$, $f(x) > f(0) = 0$.

So, $\int_0^x f(t)$ is positive and increasing.

DPP 8.5

Subjective Type

1. $x^{n-1} - x^n + x^{2n-1}$

$= x^{n-1}(1-x) + x^{2n-1} > 0$ for $x \in (0, 1)$

$\Rightarrow \quad \int_0^1 \frac{x^{n-1} - x^n + x^{2n-1}}{1+x^n}dx > 0$

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$$\Rightarrow \int_0^1 \left\{\frac{1}{1+x^n} - (1-x^{n-1})\right\} dx > 0$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^n} > \int_0^1 (1-x^{n-1})\, dx$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^n} > 1 - \frac{1}{n}\ ; \forall n \in N$$

Single Correct Answer Type

2. (c) $I_n = \int_0^{\pi/2} (\sin x + \cos x)^{n-1} (\sin x + \cos x)\, dx$

$= [(\sin x + \cos x)^{n-1} (\sin x - \cos x)]_0^{\pi/2}$
$+ \int_0^{\pi/2} (\sin x + \cos x)^{n-2} (\cos x - \sin x)^2 dx$

$= 2 + \int_0^{\pi/2} (\sin x + \cos x)^{n-2} (2 - (\sin x + \cos x)^2)\, dx$

$n.I_n = 2 + 2(n-1)\, I_{n-2} \Rightarrow n.\, I_n - 0\ 2(n-1)\, I_{n-2} = 2$

3. (b) $I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx = \int_0^1 x^{n-1}\, x\sqrt{1-x^2}\, dx$

$= -x^{n-1} \frac{(1-x^2)^{3/2}}{3}\Big|_0^1 + \int_0^1 (n-1)\, x^{n-2} \frac{(1-x^2)^{3/2}}{3} dx$

$= 0 + \frac{n-1}{3} \int_0^1 x^{n-2} (1-x^2) \sqrt{1-x^2}\, dx$

$= \frac{n-1}{3} I_{n-2} - \frac{n-1}{3} I_n$

$\Rightarrow \quad 3I_n + (n-1)\, I_n = (n-1)\, I_{n-2}$

$\Rightarrow \quad \frac{I_n}{I_n - 2} = \frac{n-1}{n+2}$

$\Rightarrow \quad \lim_{n\to\infty} \frac{I_n}{I_{n-2}} = 1.$

4. (c)

$I_n = \int_0^\infty x^n e^{-ax} dx$

$= \left[x^n \cdot \frac{e^{-ax}}{-a}\right]_0^\infty - \int_0^\infty nx^{n-1} \cdot \frac{e^{-ax}}{-a} dx$

$= -\frac{1}{a} \lim_{x\to\infty} \frac{x^n}{e^{ax}} + \frac{n}{a} I_{n-1}$

$I_n = \frac{n}{a} I_{n-1} \qquad \left[\because \lim_{x\to\infty} \frac{x^n}{e^{ax}} = 0\right]$

$= \frac{n}{a} \cdot \frac{n-1}{a} I_{n-2}$

$= \frac{n(n-1)(n-2)}{a^3} I_{n-3}$

$= \frac{n!}{a^n} \int_0^\infty e^{-ax} dx = \frac{n!}{a_n} I_0 = \frac{n!}{a^{n+1}}$

5. (a) Let $I(a) = \int_0^1 |x^n - a^n|\, dx$

$\therefore \quad I(a) = \int_0^a (a^n - x^n) dx + \int_a^1 (x^n - a^n) dx = \frac{2n}{n+1} \cdot a^{n+1} - a^n + \frac{1}{n+1}$

$\Rightarrow \quad \frac{d}{da}(I(a)) = n(2a-1)a^{n-1} \Rightarrow$ only critical point of in (0,1) is $a = \frac{1}{2} \Rightarrow I'(a) < 0$ for $a \in \left(0, \frac{1}{2}\right)$

and $I'(a) > 0$ for $a \in \left(\frac{1}{2}, 1\right)$, $I(a)$ is minimum for $a = \frac{1}{2}$.

6. (d) $f'(\ln x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ x & \text{for } x > 1 \end{cases}$

Put $\log x = t$

$\Rightarrow \quad x = e^t$

For $x > 1$; $f'(t) = e^t, t > 0$

integrating $f(t) = e^t + C$;

$f(0) = e^0 + C$

$\Rightarrow \quad C = -1$ (given $f(0) = 0$)

$\therefore \quad f(t) = e^t - 1$ for $t > 0$ (corresponding to $x > 1$)

hence $f(x) = e^x - 1$ for $x > 0$ (1)

again for $0 < x \le 1$

$f'(\log x) = 1$ $(x = e^t)$

$f'(t) = 1$ for $t \le 0$

$f(t) = t + C$

$f(0) = 0 + C$

$\Rightarrow \quad C = 0$

$\Rightarrow \quad f(t) = t$ for $t \le 0$

$\Rightarrow \quad f(x) = x$ for $x \le 0$

7. (c) Let

$I_{25} = \int_0^\infty e^{-x} \sin^{25} x\, dx$

$= (-e^{-x} \sin^{25} x)|_0^\infty + 25 \int_0^\infty e^{-x} \sin^{24} x \cdot \cos x\, dx$

$= 0 + 25 \int_0^\infty e^{-x} \sin^{24} x \cdot \cos x\, dx$

$= (-e^{-x} \sin^{24} x \cos x)\Big|_0^\infty + 25 \int_0^\infty e^{-x} (24 \sin^{23} x \cos^2 x - \sin^{25} x)\, dx$

$= 0 + 25 \int_0^\infty e^{-x} (24 \sin^{23} x - 25 \sin^{25} x)\, dx$

$= 600 I_{23} - 625 I_{25}$

$\therefore \quad 626 \frac{I_{25}}{I_{23}} = 600$

8. (d) $y = f(x)$

$\Rightarrow \quad x = f^{-1}(y) = g(y)$

$dy = f'(x)\, dx$

where y is 2 then $x = 1$ and y is 10 then $x = 5$

$\therefore \quad I = \int_1^5 f(x) dx + \int_2^{10} g(y) dy$

$= \int_1^5 f(x) dx + \int_1^5 x f'(x) dx$

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$$\therefore \quad I = \int_1^5 (f(x) + xf'(x))\,dx$$

$$= x f(x)\Big|_1^5 = 5f(5) - f(1) = 5 \cdot 10 - 2 = 48$$

9. (b) Putting $x = f(t) \Rightarrow dx = f'(t)\,dt$

$$I = \int_\pi^{2\pi} t \cdot f'(t)\,dt \quad (\because f(\pi) = \pi \text{ and } f(2\pi) = 2\pi)$$

$$= \left|t \cdot f(t)\right|_\pi^{2\pi} - \int_\pi^{2\pi} 1 \cdot f(t)\,dt$$

$$= 2\pi f(2\pi) - \pi f(\pi) - \left|\frac{t^2}{2} - \cos t\right|_\pi^{2\pi}$$

$$= 4\pi^2 - \pi^2 - \frac{1}{2}(4\pi^2 - \pi^2) + (\cos 2\pi - \cos \pi) = \frac{3\pi^2}{2} + 2$$

10. (c) Let $f^{-1}(x) = u$

$$\therefore \quad \int_{f(a)}^{f(b)} 2x(b - f^{-1}(x))\,dx$$

$$= \int_a^b 2f(u)(b-u)f'(u)\,du$$

$$= \left|(b-u)f^2(u)\right|_a^b + \int_a^b f^2(u)\,du$$

$$= \int_a^b (f^2(x) - f^2(a))\,dx$$

11. (a) Let $f(x) = 2x^3 - 9x^2 + 12x + 4$,

$\Rightarrow \quad f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2) \le 0$ for $x \in [1, 2]$.

$\Rightarrow \quad f(x)$ is decreasing in $[1, 2]$

$\therefore \quad f(2) < f(x) < f(1)$

$\therefore \quad 8 < f(x) < 9$

$$\therefore \quad \frac{1}{3} < \frac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}} < \frac{1}{\sqrt{8}}$$

$$\Rightarrow \quad \frac{1}{3}\int_1^2 dx < \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}} < \frac{1}{\sqrt{8}}\int_1^2 dx$$

$$\Rightarrow \quad \frac{1}{3} < I < \frac{1}{\sqrt{8}}$$

12. (c)

$$h(x) = \frac{g^2(x)}{2} + 3x^3 - 5$$

$$h'(x) > 0$$

$$\Rightarrow \quad g(x)\,g'(x) > -9x^2$$

$$\Rightarrow \quad \int_0^1 g(x)g'(x)\,dx > -\int_0^1 9x^2\,dx$$

$$\Rightarrow \quad \frac{(g(1))^2 - (g(0))^2}{2} > -3(1-0)$$

$$\Rightarrow \quad (g(1))^2 - 16 > -6$$

$$\Rightarrow \quad (g(1))^2 > 10$$

$$\int_{-1}^0 g(x)g'(x)\,dx > -\int_{-1}^0 9x^2\,dx$$

$$\Rightarrow \quad \frac{(g(0))^2 - (g(-1))^2}{2} > -3(0 - (-1))$$

$$\Rightarrow \quad 16 - (g(-1))^2 > -6$$

$$\Rightarrow \quad (g(-1))^2 < 22$$

$$h(5) > h(0)$$

$$\Rightarrow \quad h(5) > \frac{g^2(0)}{2} + 3(0) - 5 = 3$$

Comprehension Type

13. (a) $g(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dt$

Put $x = \dfrac{1}{1+y}$

$$\Rightarrow \quad g(m, n) = \int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1 - \frac{1}{1+y}\right)^{n-1}\left(-\frac{1}{(1+y)^2}\right)dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

14. (c) Putting $\log_e \dfrac{1}{x} = t$

$$\Rightarrow \quad x = e^{-t}$$

$$\Rightarrow \quad \int_0^1 x^m \left(\log_e \frac{1}{x}\right)^n dx$$

$$= \int_\infty^0 e^{-mt}\,t^n(-e^{-t})\,dt$$

$$= \int_0^\infty t^n\,e^{-(m+1)t}\,dt$$

$$= \frac{1}{(m+1)^{n+1}}\int_0^\infty t^n e^{-y}\,dy \quad \text{(putting } (m+1)t = y\text{)}$$

$$= \frac{f(n+1)}{(m+1)^{n+1}}$$

15. (a) $I = \int_0^1 \dfrac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= I_1 + I_2$$

In I_2, put $x = \dfrac{1}{t}$, then $I_2 = \int_\infty^1 \dfrac{\frac{1}{t^{n-1}}}{\left(1 + \frac{1}{t}\right)^{m+n}}\left(\frac{-dt}{t^2}\right)$

$$= \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore \quad I = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = g(m, n)$$

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CHAPTER 9

DPP 9.1

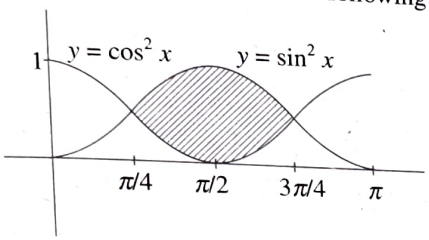
Single Correct Answer Type

1. (c) $y = \sin^2 x$ and $y = \cos^2 x$

Solving $\sin^2 x = \cos^2 x$

$\therefore \quad x = \pi/4, 3\pi/4$

Graph of functions is as shown in the following figure.



From the figure, the required area

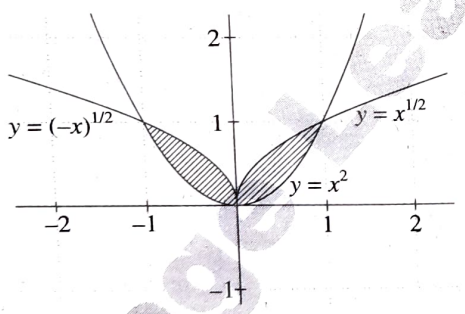
$$= \int_{\pi/4}^{3\pi/4} (\sin^2 x - \cos^2 x)\, dx$$

$$= -\int_{\pi/4}^{3\pi/4} \cos 2x\, dx = 1$$

2. (b) $y = x^2$ is upward parabola

$$y = \sqrt{|x|} = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

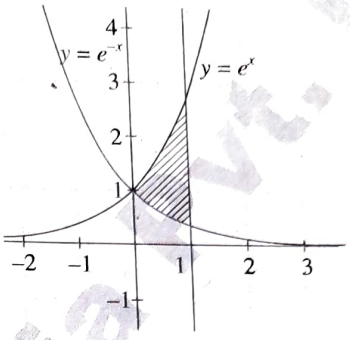
The graph of the functions is as shown in the following figure:



$$A = 2\int_0^1 \left(\sqrt{x} - \frac{x^2}{2}\right) dx$$

$$= \frac{2}{3}$$

3. (d) The graph of the functions is as shown in the following figure:



$$A = \int_0^1 (e^x - e^{-x})\, dx$$

$$= (e^x + e^{-x})_0^1$$

$$= (e + e^{-1}) - (1 + 1)$$

$$= e + \frac{1}{e} - 2$$

4. (b) We have $y = |\cos^{-1}(\sin x)| - |\sin^{-1}(\cos x)|$

$$= \left|\frac{\pi}{2} - \sin^{-1}(\sin x)\right| - \left|\frac{\pi}{2} - \cos^{-1}(\cos x)\right|$$

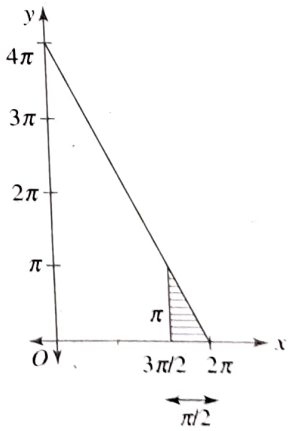
$$= \left|\frac{\pi}{2} - (x - 2\pi)\right| - \left|\frac{\pi}{2} - (2\pi - x)\right|$$

$$= \left|\frac{5\pi}{2} - x\right| - \left|x - \frac{3\pi}{2}\right|$$

$$= \left(\frac{5\pi}{2} - x\right) - \left(x - \frac{3\pi}{2}\right)$$

$$= 4\pi - 2x$$

The graph of the function is as shown in the following figure.



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From the figure, Required area
= Area of shaded triangle in figure

$$= \frac{1}{2} \times \left(\frac{\pi}{2}\right) \times \pi = \frac{\pi^2}{4}$$

5. (a) $(a, 0)$ lies on the given curve

$\therefore \quad 0 = \sin 2a - \sqrt{3} \sin a$

$\Rightarrow \quad \sin a = 0$ or $\cos a = \sqrt{3}/2$

$\Rightarrow \quad a = \frac{\pi}{6}$ (as $a > 0$ and the first point of intersection with positive X-axis)

$$\text{Required area } A = \int_0^{\pi/6} (\sin 2x - \sqrt{3} \sin x)\,dx$$

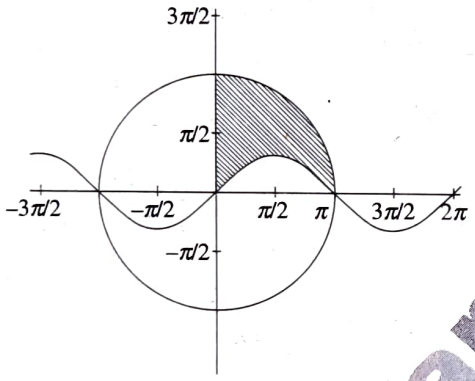
$$= \left(-\frac{\cos 2x}{2} + \sqrt{3} \cos x\right)_0^{\pi/6}$$

$$= \left(-\frac{1}{4} + \frac{3}{2}\right) - \left(-\frac{1}{2} + \sqrt{3}\right)$$

$$= \frac{7}{4} - 2 \cos a$$

$\Rightarrow \quad 4A + 8 \cos a = 7$

6. (a) Graph of the functions is as shown in the following figure:

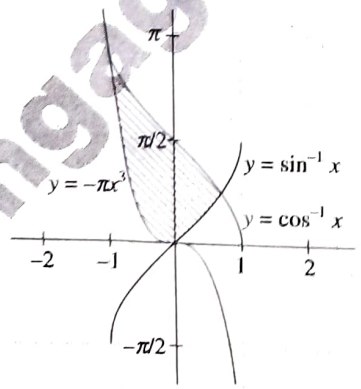


From the figure bounded area A
= (Area of quarter of circle) − (Area bounded by $y = \sin x$ and x-axis between $x = 0$ and $x = \pi/2$)

$$= \frac{\pi(\pi)^2}{4} - 2$$

$$= \frac{(\pi^3 - 8)}{4}$$

7. (d) Graph of the functions is as shown in the following figure:



From the figure, bounded area

$$A = \left|\int_0^{\pi} \left(\frac{-y}{\pi}\right)^{1/3} dy\right| - \left|\int_{\pi/2}^{\pi} \cos y\,dy\right| + \left|\int_{\pi/4}^{\pi/2} \cos y\,dy\right| + \left|\int_0^{\pi/4} \sin y\,dy\right|$$

$$= \left|\frac{-3y^{4/3}}{4\pi^{1/3}}\right|_0^{\pi} - |\sin y|_{\pi/2}^{\pi} + |\sin y|_{\pi/4}^{\pi/2} + |-\cos y|_0^{\pi/4}$$

$$= \left|-\frac{3}{4}\pi\right| - |-1| + \left|1 - \frac{1}{\sqrt{2}}\right| + \left|-\frac{1}{\sqrt{2}} + 1\right|$$

$$= \frac{3}{4}\pi - 1 + 1 - \frac{1}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}}$$

$$= \frac{3\pi}{4} + 1 - \sqrt{2}$$

8. (b) $y = \sin^2 x - 2 \sin x = \sin x (\sin x - 2)$

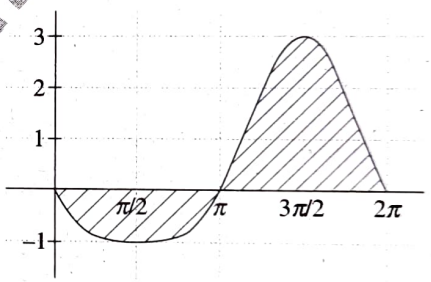
$y = 0 \therefore \sin x = 0 \therefore x = 0, \pi, 2\pi$

Also $\sin x - 2 < 0$

$\therefore \quad y > 0$ if $\sin x < 0$

and $\quad y < 0$ if $\sin x > 0$

$\therefore$ Graph of the function is shown in the figure:



The required area = The shaded area

$$= \left|\int_0^{\pi} (\sin^2 x - 2 \sin x)dx\right| + \int_{\pi}^{2\pi} (\sin^2 x - 2 \sin x)dx$$

$$= \left|\frac{1}{2}\int_0^{\pi} (1 - \cos 2x)dx - 2\int_0^{\pi} \sin x\,dx\right| + \int_{\pi}^{2\pi} \left(\frac{1 - \cos 2x}{2} - 2 \sin x\right)dx$$

$$= \left|\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + 2 \cos x\right|_0^{\pi} + \left[\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + 2 \cos x\right]_{\pi}^{2\pi}$$

$$= \left|\frac{1}{2}\pi - 4\right| + \frac{2\pi - \pi}{2} + 4$$

$$= \left(4 - \frac{\pi}{2}\right) + 4 + \frac{\pi}{2} = 8 \text{ sq. units}$$

9. (a) $y = 2x - x^2$ is downward parabola intersecting x-axis at $x = 0$ and $x = 2$

Solving $f(x) = 2x - x^2$ and $g(x) = x^n$

we have $2x - x^2 = x^n$

$\Rightarrow \quad x = 0$ and $x = 1$

$$A = \int_0^1 (2x - x^2 - x^n)dx = \left[x^2 - \frac{x^3}{3} - \frac{x^{n+1}}{n+1}\right]_0^1$$

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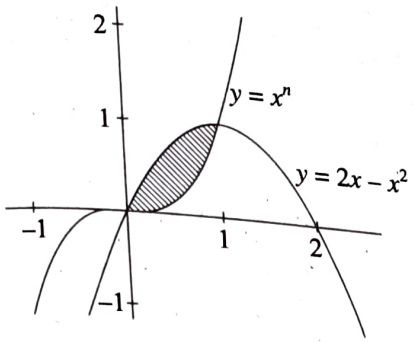


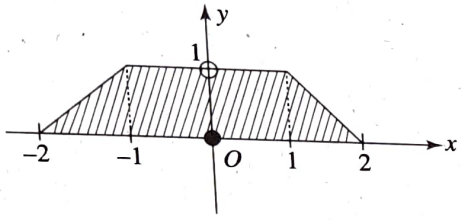
$$= 1 - \frac{1}{3} - \frac{1}{n+1} = \frac{2}{3} - \frac{1}{n+1}$$

hence, $\frac{2}{3} - \frac{1}{n+1} = \frac{1}{2} \Rightarrow \frac{2}{3} - \frac{1}{2} = \frac{1}{n+1}$

$\Rightarrow \quad n = 5$

10. (b) $f(x) = \begin{cases} \{x\}, & -2 \le x < -1 \\ |\text{sgn } x|, & -1 \le x \le 1 \\ \{-x\}, & 1 < x \le 2 \end{cases}$

The graph of $f(x)$ is



$$\text{Area} = \frac{2+4}{2} \times 1 = 3 \text{ sq. units}$$

11. (b) Solving given curves

$$x^2 + 2 = 2|x| - \cos \pi x$$

$$\Rightarrow \quad x^2 - 2|x| + 2 = -\cos \pi x$$

$$\Rightarrow \quad (|x| - 1)^2 + 1 = -\cos \pi x$$

$$\Rightarrow \quad x = \pm 1$$

$$\therefore \text{ Required Area} = \int_{-1}^{1} (x^2 + 2 - 2|x| + \cos \pi x)\, dx = 8/3$$

12. (b) $y = f(x) = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$

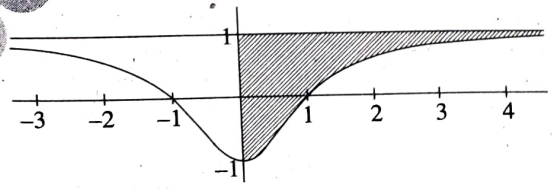
$y = 0, \quad \therefore \quad \frac{x^2-1}{x^2+1} = 0 \quad$ or $\quad x = \pm 1$

Also function is even.

$y(0) = -1$ is the minimum value of function.

$$\lim_{x \to \pm\infty} \left(1 - \frac{2}{x^2+1}\right) = 1$$

The graph of the function is as shown in the following figure:



$$A = 2\int_0^{\infty} \left(1 - \frac{x^2-1}{x^2+1}\right) dx$$

$$= 4\int_0^{\infty} \frac{dx}{x^2+1}$$

$$= 4 \cdot \tan^{-1}|_0^{\infty} = 4 \cdot \frac{\pi}{2} = 2\pi$$

13. (a) $y = xe^{-x}$

$$y' = e^{-x} - x\,e^{-x} = (1-x)e^{-x}$$

y increases for $x < 1$ and decreases for $x > 0$

$x = 1$ is point maxima.

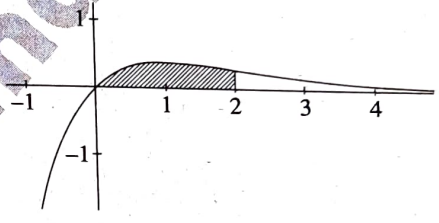
Also $\lim_{x \to \infty} xe^{-x} = 0$

$$y'' = -e^{-x} - [e^{-x} - xe^{-x}]$$

$$= e^{-x}[-1 - 1 + x]$$

$$= (x-2)e^{-x}$$

For point of inflection $y'' = 0 \Rightarrow x = 2$



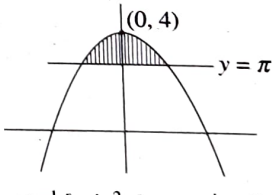
$$\therefore \text{ Required area is } A = \int_0^2 xe^{-x}\, dx = -xe^{-x}|_0^2 + \int_0^2 e^{-x} dx$$

$$= (-2e^{-2}) - (e^{-x})_0^2$$

$$= 1 - 3e^{-2}$$

14. (c) $0 \le \sin^2 x \le 1 \Rightarrow [-\sin^2 x] = 0$ or -1

But $\sec^{-1}(0)$ is not defined.



$$\Rightarrow \quad y = \sec^{-1}[-\sin^2 x] = \sec^{-1}(-1) = \pi$$

Solving with $y = \frac{16 - x^2}{4}$, $16 - x^2 = 4\pi$

$$\therefore \quad x = \pm 2\sqrt{4 - \pi}$$

$$\text{Required area} = \int_{-2\sqrt{4-\pi}}^{2\sqrt{4-\pi}} \left(\frac{16 - x^2}{4} - \pi\right) dx = \frac{8}{3}(4 - \pi)^{3/2}$$

Multiple Correct Answers Type

15. (b, d)

$$I = \int (6x - 3x^2) dx = \frac{6x^2}{2} - \frac{3x^3}{3} = 3x^2 - x^3 = x^2(3 - x)$$

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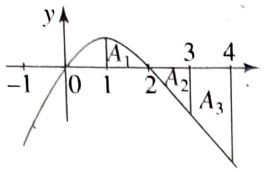


$A_1 = I(2) - I(1) = 4 - 2 = 2$ units

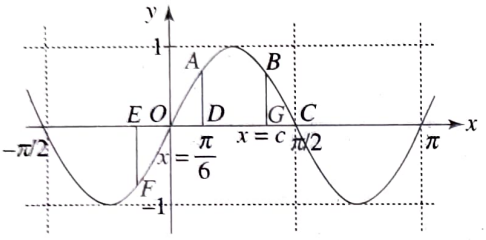
$A_2 = I(2) - I(3) = 4 - 0 = 4$ units

$A_3 = I(3) - I(4) = 0 - (-16) = 16$ units

$\Rightarrow$ one value of a will lie in (3, 4).

Using symmetry, other will lie in (−2, −1)

16. (a, b)



Area $OABC = \int_0^{\pi/2} \sin 2x\,dx = 1$

Area $OAD = \int_0^{\pi/6} \sin 2x\,dx = \frac{1}{4}$

$\because$ $\sin 2x$ is symmetric about origin, therefore,

$$c = -\frac{\pi}{6} \quad (\because \text{ area } OAD = \text{Area } OEF)$$

Now, $\int_{\frac{\pi}{6}}^{c} \sin 2x\,dx = \frac{1}{2}$

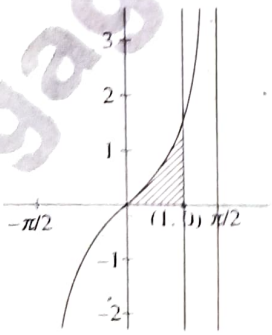
$$\cos 2c = -\frac{1}{2}$$

$\therefore$ $c = \frac{\pi}{3}$

$\therefore$ $c = -\frac{\pi}{6}, \frac{\pi}{3}$

17. (a, b)

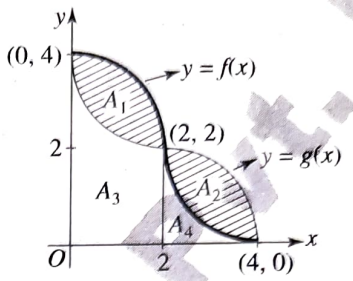
Required area $= \int_0^1 \tan x\,dx = \int_0^1 \tan(1-x)\,dx$



Also required area $= \tan 1 - \int_0^{\tan 1} \tan^{-1} y\,dy$

DPP 9.2

1. (c) Given $\int_0^4 f(x)\,dx - \int_0^4 g(x)\,dx = 10$



$\therefore$ $(A_1 + A_3 + A_4) - (A_2 + A_3 + A_4) = 10$

$\therefore$ $A_1 - A_2 = 10$ (i)

Also $\int_2^4 g(x)\,dx - \int_2^4 f(x)\,dx = 5$

$(A_2 + A_4) - A_4 = 5$

$A_2 = 5$ (ii)

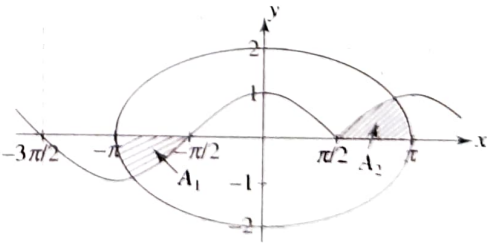
$\therefore$ Adding (i) and (ii), $A_1 = 15$

2. (a) C_1: $4x^2 + \pi^2 y^2 = 4\pi^2$

$\therefore$ C_1: $\frac{x^2}{\pi^2} + \frac{y^2}{4} = 1$

Curve C_2: $y = -\text{sgn}\left(x - \frac{\pi}{2}\right)\cos x$

$$= \begin{cases} -\cos x, & x > \frac{\pi}{2} \\ 0, & x = \frac{\pi}{2} \\ \cos x, & x < \frac{\pi}{2} \end{cases}$$



Area A_1 and A_2 are equal.

$\Rightarrow$ Required ratio $= \frac{\pi^2 - 2}{\pi^2 + 2}$

3. (c) Rotating the system through an angle 60° in clockwise direction, the equations in the question become

$y = \log_e x,\ y = 0,\ x = 2$

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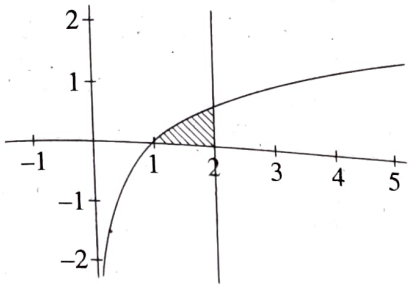


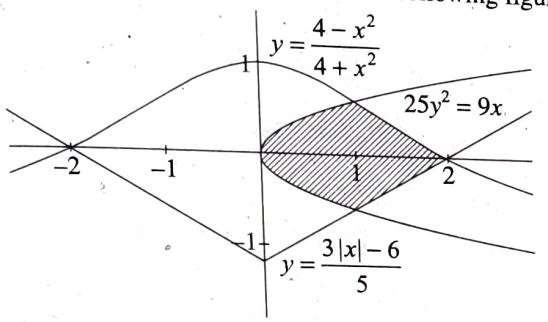
Since after rotation the area remains unchanged

Required area $= \int_1^2 (\log_e x)dx$

$= [x \log_e x - x]_1^2 = [2 \log_e 2 - 2] + 1$

$= 2 \log_e 2 - 1$ sq. units

4. (a) Graph of the functions is shown in the following figure



Required area $= 2\int_0^1 \frac{3}{5}\sqrt{x}dx + \int_1^2 \left\{\frac{4-x^2}{4+x^2} - \frac{3x-6}{5}\right\}dx$

$= \frac{4}{5} + \int_1^2 \left\{\frac{8}{4+x^2} - \frac{3x-1}{5}\right\}dx$

$= \frac{4}{5} + \left[4 \tan^{-1}\frac{x}{2}\right]_1^2 - \frac{1}{5}\left[\frac{3x^2}{2} - x\right]_1^2$

$= \frac{4}{5} + \pi - 4 \tan^{-1}\frac{1}{2} - \frac{7}{10}$

$= \left\{\pi - 4 \tan^{-1}\frac{1}{2} + \frac{1}{10}\right\}$ sq. units

5. (a)

Min. $\{|x|, |y|\} = 1$

$\therefore |x| = 1$ if $|x| < |y|$, graph of which is two parallel lines $x = \pm 1$ excluding for $-1 < y < 1$

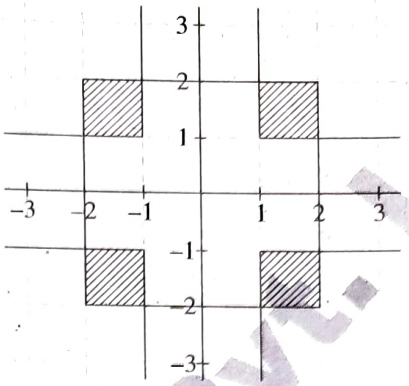
and $|y| = 1$ if $|x| > |y|$, graph of which is two parallel lines $y = \pm 1$ excluding for $-1 < x < 1$

Max. $\{|x|, |y|\} = 2$

$\therefore |x| = 2$ if $|x| > |y|$, graph of which is two parallel lines $x = \pm 2$ for $-2 < y < 2$

and $|y| = 2$ if $|y| > |x|$, graph of which is two parallel lines $y = \pm 2$ for $-2 < x < 2$

Thus, graph of the functions is as shown in the following figure.

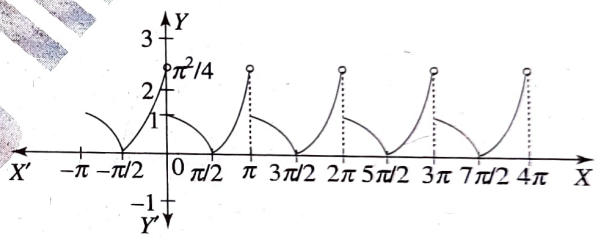


As shown in the figure area $= 4 \times 1 \times 1 = 4$

6. (d)

Given $f(x) = \begin{cases} \cos x & 0 \le x < \frac{\pi}{2} \\ \left(\frac{\pi}{2} - x\right)^2 & \pi/2 \le x < \pi \end{cases}$ and f is periodic with period π

Graph of the function is as shown in the following figure.



From the graph, the range of the function is $\left[0, \frac{\pi^2}{4}\right)$

It is discontinuous at $x = n\pi$, $n \in I$.

Area bounded by $y = f(x)$ and the X-axis from 0 to $n\pi$ for $n \in N$

$= n\int_0^{\pi} f(x)dx$

$= n\left[\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} \left(\frac{\pi}{2} - x\right)^2 dx\right] = n\left(1 + \frac{\pi^3}{24}\right)$

7. (c)

$f(x) = [x] + \sqrt{x - [x]}$

$= \begin{cases} 0 + \sqrt{x}, & 0 \le x < 1 \\ 1 + \sqrt{x-1}, & 1 \le x < 2 \\ 2 + \sqrt{x-2}, & 2 \le x < 3 \\ \ldots \\ \ldots \end{cases}$

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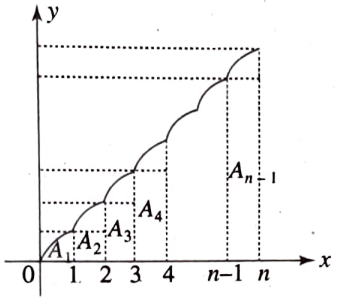
The graph of the function is as shown in the following figure:



Area for $0 \le x < 1$, $\int_0^1 \sqrt{x}\,dx$

Area for $1 \le x < 2$, $\int_0^1 \sqrt{x}\,dx + 1 \times 1$

Area for $2 \le x < 3$, $\int_0^1 \sqrt{x}\,dx + 2 \times 1$

Area for $n-1 \le x < n$, $\int_0^1 \sqrt{x}\,dx + (n-1) \times 1$

So total area $= n\int_0^1 \sqrt{x}\,dx + [1 + 2 + 3 + \ldots + (n-1)]$

$$= \frac{2n}{3} + \frac{n(n-1)}{2}$$

8. (d) $x^2 + y^2 \le 100$

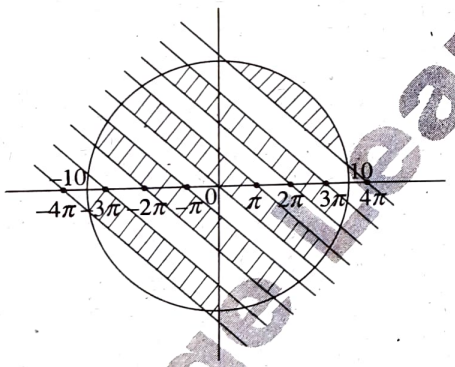
$\Rightarrow$ Points lies inside circle or on the circle.

$\sin(x + y) > 0$

$x + y \in (0, \pi) \cup (2\pi, 3\pi)\ldots$

$x + y = c$ is equation of a line

Required area is shown in the following figure:



Required area = Shaded region

= Half of the area of the circle

$$= \frac{1}{2}\pi(10)^2 = 50\pi$$

9. (b) The region is enclosed between two parallelograms of areas 4×2 and 2×1

The area $= 8 - 2 = 6$.

10. (b) $f(x)$ and $g(x)$ are periodic with period 1.

$\therefore$ Required area is $\int_0^6 [f(x) - g(x)]dx = 6\int_0^1 (\sqrt{x} - x^2)dx = 2$

11. (b)

$$\frac{xy(x^2 - 3x + 2)}{x^2 - 7x + 12} > 0$$

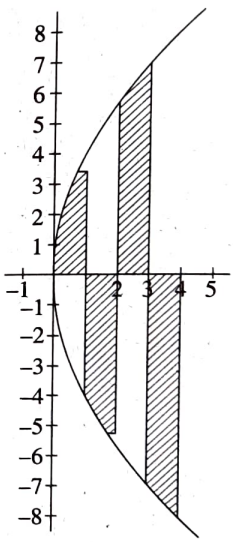
$$\Rightarrow \quad \frac{xy(x-1)(x-2)}{(x-3)(x-4)} > 0$$

$$\Rightarrow \quad \begin{cases} y > 0, & \text{if } \dfrac{x(x-1)(x-2)}{(x-3)(x-4)} > 0 \\ y < 0, & \text{if } \dfrac{x(x-1)(x-2)}{(x-3)(x-4)} < 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} y > 0, & \text{if } x \in (0,1) \cup (2,3) \cup (4,\infty) \\ y < 0, & \text{if } x \in (-\infty, 0) \cup (1,2) \cup (3,4) \end{cases}$$

$y^2 < 16x$ is interior of the parabola $y^2 = 16x$

Region is as shown in the following figure:



From the figure, required area

= Half of the area of the region bounded by

$y^2 = 16x$ and $x = 4$

$$= \frac{1}{2}\left(2\int_0^4 4\sqrt{x}\,dx\right) = \frac{64}{3}$$

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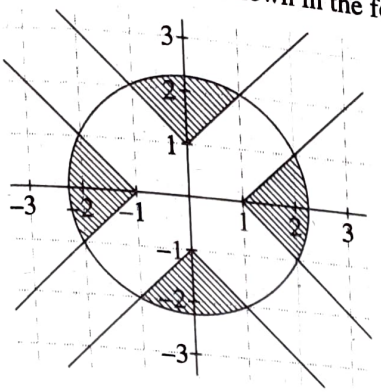
12. (b) Graph of the inequalities is as shown in the following figure:



Solving $y = x - 1$ with the circle $x^2 + y^2 = 5$ we get

$x^2 + (x-1)^2 = 5$ or $x = 2$

Required area

$$= 8\left(\frac{1}{2}\cdot 1.1 + \int_2^{\sqrt{5}} \sqrt{5-x^2}\,dx\right)$$

$$= 4 + 8\left(\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right)_2^{\sqrt{5}}$$

$$= 4 + 8\left(0 + \frac{5\pi}{4} - 1 - \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

$$= 10\pi - 4 - 20\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

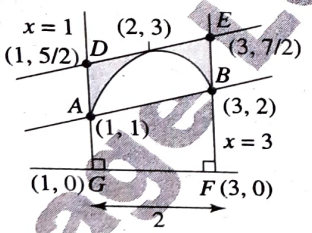
$$= 20\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) - 4$$

$$= 20\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) - 4$$

$$= 20\tan^{-1}\left(\frac{1}{2}\right) - 4$$

$$= 5\left(\pi - \tan^{-1}\left(\frac{24}{7}\right)\right) - 4 \quad \left(\because 4\tan^{-1}\left(\frac{1}{2}\right) = \pi - \tan^{-1}\left(\frac{24}{7}\right)\right)$$

13. (b)



Equation of L_2: $y - 1 = \frac{2-1}{3-1}(x-1)$

or $2y - 2 = x - 1$

$2y - x = 1$

$\Rightarrow$ Slope of $L_2 = 1/2$

Equation of $L_1 = y - 3 = \frac{1}{2}(x-2)$

or $2y - x = 4$

$\therefore \quad D\left(1, \frac{5}{2}\right)$ and $E\left(3, \frac{7}{2}\right)$

Area under $f(x) = 4$

$\therefore$ Shaded area = Area of trapezium $DEFG$ – Area under $f(x)$

$$= \frac{1}{2}\left(\frac{5}{2} + \frac{7}{2}\right) \times 2 - 4$$

$$= 6 - 4 = 2$$

Comprehension Type

14. (b), 15. (c)

$$A_1 = \int_0^a (\sin x - f(x))dx = 1 + (a-1)\cos a - \sin a$$

Differentiating w.r.t. 'a' both sides.

$\therefore \quad \sin a - f(a) = \cos a - (a-1)\sin a - \cos a$

$\therefore \quad f(a) = a\sin a$

$\therefore \quad f(x) = x\sin x$

Solving $y = \sin x$ with $y = x\sin x$, we have $x\sin x = \sin x$

$\therefore \quad x = 1$ or $x = \pi, 2\pi$

$\therefore \quad A_2 = \int_1^{\pi} (x\sin x - \sin x)dx$

$$= (-x\cos x)_1^{\pi} - \int_1^{\pi} -\cos x\,dx + (\cos x)_1^{\pi}$$

$$= (\pi + \cos 1) + (0 - \sin 1) + (-1 - \cos 1)$$

$$= (\pi - \sin 1 - 1) \text{ sq. units}$$

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CHAPTER 10

DPP 10.1

Single Correct Answer Type

1. (a) $y = c_1 \cos 2x + c_2 \cos^2 x + c_3 \sin^2 x + c_4$

$= c_1 \cos 2x + \frac{c_2}{2}(1 + \cos 2x) + \frac{c_3}{2}(1 - \cos 2x) + c_4$

$= \left(\frac{c_2}{2} + \frac{c_3}{2} + c_4\right) + \left(c_1 + \frac{c_2}{2} - \frac{c_3}{2}\right)\cos 2x$

$= A + B \cos 2x$

That means there are two independent parameter.

Thus, the order of differential equation will be 2.

2. (b) $y = \frac{ax}{bx + c}$

$\Rightarrow \quad y = \frac{x}{\frac{b}{a}x + \frac{c}{a}}$

$\Rightarrow \quad y = \frac{x}{\lambda x + \mu}$

$\Rightarrow$ Order is 2

3. (d) $y^2\left(\frac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x = -3x\left(\frac{dy}{dx}\right)^{\frac{1}{3}}$

$\therefore \quad \left(y^2\left(\frac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x\right)^3 = -27x^3\left(\frac{dy}{dx}\right)$

Here order = 2 = p

Degree = 6 = q

$\therefore \quad p < q$

4. (d) Multiply by y'''

$\therefore$ Order = 3, degree = 2

5. (d) $(px + g)\,dx + (3y + f)\,dy = 0$

Integrating $\frac{px^2}{2} + gx + \frac{3y^2}{2} + fy + c = 0$

$\Rightarrow \quad p = 3$ (for a circle)

6. (d) $\frac{dy}{dx} = 1 - x^2 - y^2 + x^2y^2$

$\therefore \quad \frac{dy}{dx} = (1 - x^2)(1 - y^2)$

$\therefore \quad \frac{dy}{1 - y^2} = (1 - x^2)\,dx$

$\therefore \quad \int \frac{dy}{1 - y^2} = \int (1 - x^2)\,dx$

$\therefore \quad \frac{1}{2}\log\left(\frac{1 + y}{1 - y}\right) = x - \frac{x^3}{3} + C$

7. (d) $e^{-x}(y + 1)dy + (\cos^2 x - \sin 2x)\,y(dx) = 0$

$\therefore \quad (1 + y^{-1})dy + e^x(\cos^2 x - \sin 2x)dx = 0$

Integrating we get,

$y + \log_e y + e^x \cos^2 x = c$

$y = 1$ when $x = 0$

$\therefore \quad 1 + 0 + 1 = c$

$\therefore \quad c = 2$

$\therefore$ Curve is $y + \log y + e^x \cos^2 x = 2$

8. (a) The given equation

$(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0$

$\Rightarrow \quad \frac{1 - y}{y^2}dy + \frac{1 + x}{x^2}dx = 0$

$\Rightarrow \quad \left(\frac{1}{y^2} - \frac{1}{y}\right)dy + \left(\frac{1}{x^2} + \frac{1}{x}\right)dx = 0$

On integrating, we get the required solution

$\log\left(\frac{x}{y}\right) = \frac{1}{x} + \frac{1}{y} + c$

9. (d) $\frac{y_2}{y_1} = 1$

$\therefore \quad \frac{dp}{dx} = p\left(\text{where } p = \frac{dy}{dx}\right)$

$\therefore \quad \ln p = x + c$

$\therefore \quad p = e^{x + c}$

$\therefore \quad \frac{dy}{dx} = ke^x$

$\therefore \quad y = ke^x + \lambda$

Satisfying (0, 0), So $\lambda = -k$

$\therefore \quad y = k(e^x - 1)$

10. (a) $\frac{dx}{dy} + \frac{x^2}{y^2} - \frac{x}{y} + 1 = 0$

Put $x = vy$

$\Rightarrow \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$

$\therefore \quad v + y\frac{dv}{dy} + v^2 - v + 1 = 0$

$\Rightarrow \quad -y\frac{dv}{dy} = (1 + v^2)$

$\Rightarrow \quad \frac{dv}{v^2 + 1} = -\frac{dy}{y}$

Integrating

$\tan^{-1} v + C = -\ln y$

$\Rightarrow \quad \tan^{-1}\left(\frac{x}{y}\right) + \ln y + C = 0$

where C is arbitrary constant

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11. (c) $(1 - xy + x^2y^2)dx = x^2\,dy$

$(1 - (xy) + (xy)^2)dx = x^2\,dy$

Let $xy = v$.

$$y = \frac{v}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x \cdot \frac{dv}{dx} - v}{x^2}$$

$$x^2dy = xdv - v\,dx$$

So $(1 - v + v^2)dx = xdv - vdx$

$$\Rightarrow \quad (1 + v^2)dx = xdv$$

$$\Rightarrow \quad \int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \tan^{-1}v = \log|x| + \log c$$

$$\Rightarrow \quad \tan^{-1}v = \log|cx|$$

$$\Rightarrow \quad v = \tan \log|cx|$$

$$\Rightarrow \quad xy = \tan \log|cx|$$

Multiple Correct Answers Type

12. (a, b, d)

$$y = c \sin x \qquad \text{(i)}$$

$$\therefore \quad \frac{dy}{dx} = c \cos x \qquad \text{(ii)}$$

From (ii)

$$\left(\frac{dy}{dx}\right)^2 = c^2 \cos^2 x \qquad \text{(iii)}$$

Putting $c = \frac{y}{\sin x}$ from (i), $\left(\frac{dy}{dx}\right)^2 = y^2 \cot^2 x$

Eliminating c from (i) and (ii), $\frac{dy}{dx} = y \cot x$

Squaring and adding (i) and (ii), $y^2 + \left(\frac{dy}{dx}\right)^2 = c^2$.

Putting the value of 'c' from (iii), $y^2 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\sec x\right)^2$

13. (a, c)

$$x = Ae^{-3t} + Be^{-t}$$

$$\therefore \quad \frac{dx}{dt} = -3Ae^{-3t} - Be^{-t}$$

When $t = 0$, $x = 0$ (i)

$$\therefore \quad A + B = 0$$

At $t = 0$, $\frac{dx}{dt} = 12$ (ii)

$$\therefore \quad 12 = -3A - B$$

Solving, we get $A = -6$, $B = 6$.

14. (a, b)

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 - 2\left(x + \frac{1}{4x}\right)\frac{dy}{dx} + 1 = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx} - 2x\right)\left(\frac{dy}{dx} - \frac{1}{2x}\right) = 0$$

$$\therefore \quad dy = 2x\,dx \text{ or } dy = \frac{1}{2x}dx$$

$$\Rightarrow \quad y = x^2 + c \text{ or } y = \frac{1}{2}\ln x + c,\ x > 0.$$

15. (a, b)

$$\left(\frac{dy}{dx}\right)^2 + 2y\cot x\frac{dy}{dx} - y^2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = -y(\cot x \pm \operatorname{cosec} x)$$

$$\Rightarrow \quad \frac{dy}{y} = (-\cot x \pm \operatorname{cosec} x)dx$$

$$\Rightarrow \quad \log y = -[\log \sin x \pm \log(\operatorname{cosec} x + \cot x)] + \log c$$

$$= -\log \frac{\sin x}{c(\operatorname{cosec} x \pm \cot x)}$$

$$= \log \frac{c(1 \pm \cos x)}{\sin^2 x}$$

$$= \log\left(\frac{c}{1 \pm \cos x}\right)$$

$$\Rightarrow \quad y = \frac{c}{1 \pm \cos x}$$

DPP 10.2

Single Correct Answer Type

1. (d) $y'(t) + 2y(t) = 2e^{2t}$ is linear differential equation

I.F. $= e^{2t}$

$\therefore$ solution is $y\,e^{2t} = 2t + C$

Given $y(0) = 2$

$$\Rightarrow \quad 2 = C$$

$$\Rightarrow \quad ye^{2t} = 2(t + 1)$$

When $t = 1$, $y(1) = \frac{4}{e^2}$

2. (a) $\frac{dy}{dx} - y\log_e 2 = 2^{\sin x}(\cos x - 1)\log_e 2$

This is linear differential equation

I.F. $= e^{-\log_e 2\int dx} = e^{-x\log_e 2} = 2^{-x}$

Solution is

$$y\,2^{-x} = \int 2^{-x}\,2^{\sin x}\,(\cos x - 1)\log_e 2\,dx$$

put $\sin x - x = t \Rightarrow (\cos x - 1)dx = dt$

$$\therefore \quad y\,2^{-x} = \log_e 2\int 2^t\,dt$$

$$\therefore \quad y\,2^{-x} = 2^t + c$$

$$\therefore \quad y = 2^{x+t} + c2^x$$

$$\therefore \quad y = 2^{\sin x} + c2^x$$

3. (b) $ye^y\,dx = (y^3 + 2xe^y)dy$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{y^2}{e^y} + \frac{2x}{y}$$

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$$\Rightarrow \quad \frac{dx}{dy} - \frac{2}{y}x = \frac{y^2}{e^y}$$

$$\text{I.F.} = e^{-2\ln y} = \frac{1}{y^2}$$

General solution $x\ \frac{1}{y^2} = \int \frac{y^2}{e^y}\frac{1}{y^2}dy + c$

$$\frac{x}{y^2} = e^{-y} + c\ = -e^{-y} + c$$

$x = 0, y = 1 \Rightarrow c = e^{-1}$

$\Rightarrow \quad x = -y^2(e^{-y} - e^{-1})$

If $y = 0$, then $x = 0$

4. (a) $\frac{dy}{dx} - f(x).\ y = 0$

$$\therefore \quad \frac{dy}{y} = f(x)\,dx$$

$$\therefore \quad \ln y = \int f(x)\,dx$$

$$\therefore \quad y_1(x) = e^{\int f(x)dx}.$$

Then for given equation, I.F. $= e^{\int f(x)dx}$

Hence, solution is $y.y_1(x) = \int r(x)\cdot y_1(x)\,dx + c$

$$y = \frac{1}{y_1(x)}\int r(x)\cdot y_1(x)dx + \frac{c}{y_1(x)}$$

5. (a) $\frac{dy}{dx} + \frac{\log x}{x}y = x^{-\frac{1}{2}\log x}$

$$\text{I.F} = e^{\int \frac{\log x}{x}dx} = e^{\frac{(\log x)^2}{2}} = (e^{\log x})^{\frac{\log x}{2}} = x^{\frac{1}{2}\log x}$$

$$\therefore \quad \text{Solution is } x^{\frac{1}{2}\log x}\cdot y = \int dx$$

$$\text{or} \quad yx^{\frac{1}{2}\log x} = x + c$$

6. (b) We have, $(1 + \tan y)(dx - dy) + 2x\,dy = 0$

$$\Rightarrow \quad (1 + \tan y)dx = (1 + \tan y - 2x)dy$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{2}{1+\tan y}x = 1, \text{ which is linear differential equation.}$$

$$\text{I.F.} = e^{2\int \frac{dy}{1+\tan y}} = e^{\int \frac{2\cos y}{\sin y+\cos y}dy}$$

$$= e^{\int\left(1+\frac{\cos y-\sin y}{\sin y+\cos y}\right)dy} = e^{y+\log(\cos y+\sin y)}$$

$$= (\cos y + \sin y)e^y$$

So, the solution is

$xe^y(\sin y + \cos y) = \int e^y\ (\sin y + \cos y)dy + c$

i.e. $xe^y(\sin y + \cos y) = e^y \sin y + c.$

i.e. $x(\sin y + \cos y) = \sin y + ce^{-y}.$

7. (c) $x^2 y - x^3\frac{dy}{dx} = y^4 \cos x$

$$\frac{1}{y^4}\frac{dy}{dx} + \left(\frac{-1}{x}\right)\frac{1}{y^3} = -\frac{\cos x}{x^3}$$

put $\frac{1}{y^3} = v$

$$\frac{1}{y^4}\frac{dy}{dx} = -\frac{1}{3}\frac{dv}{dx}$$

$$\frac{-1}{3}\frac{dv}{dx} + \left(-\frac{-1}{x}\right)v = -\frac{\cos x}{x^3}$$

$$\frac{dv}{dx} + \frac{3}{x}v = 3\cos x$$

$$\text{I.F.} = e^{3\int \frac{1}{x}dx} = x^3$$

solution $v.\ x^3 = 3\int \cos x\,dx + c$

$$\frac{1}{y^3}x^3 = 3\sin x + c$$

$$x^3y^{-3} = 3\sin x + c$$

8. (d) $\frac{dx}{dy} = xy^3 + x^2y^7$

$$\Rightarrow \quad \frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y^3 = y^7$$

$$\Rightarrow \quad \frac{d(-1/x)}{dy} - \frac{1}{x}(y^3) = y^7$$

$$\text{I.F.} = e^{y^4/4}$$

Solution is $\frac{-e^{y^4/4}}{x} = y^4e^{y^4/4} - 4e^{y^4/4} - e^{1/4}$

when $y = -1, x = \frac{1}{4}$

$$\therefore \quad \frac{dy}{dx} = -\frac{16}{5}$$

9. (c) We have $\frac{dy}{dx} = y\tan x - y^2\sec x$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$$

Putting $\frac{1}{y} = v \Rightarrow \frac{-1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$, we get

$\frac{dv}{dx} + \tan x\ .\ v = \sec x$ which is linear diff. equation

$$\text{I.F.} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

$\therefore$ The solution is

$v\sec x = \int \sec^2 x dx + c$

$$\Rightarrow \quad \frac{1}{y}\sec x = \tan x + c$$

$$\Rightarrow \quad \sec x = y(c + \tan x)$$

Multiple Correct Answers Type

10. (a, b, d)

$\frac{dy}{dx} + y = f(x)$ is linear differential equation.

I.F. $= e^x$

$\therefore$ Solution is $ye^x = \int e^x f(x)\,dx + C$

now if $0 \le x \le 2$ then $ye^x = \int e^x e^{-x}\,dx + C$

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$\Rightarrow \quad ye^x = x + C$

$y(0) = 1, \Rightarrow C = 1$

$\therefore \quad ye^x = x + 1$

$\therefore \quad y = \dfrac{x+1}{e^x}$; (i)

$\therefore \quad y(1) = \dfrac{2}{e}$

Also $y' = \quad y' = \dfrac{e^x - (x+1)e^x}{e^{2x}}$

$\Rightarrow \quad y'(1) = \dfrac{e-2e}{e^2} = \dfrac{-e}{e^2} = -\dfrac{1}{e}$

If $x > 2$

$$ye^x = \int e^{x-2}dx$$

$\therefore \quad ye^x = e^{x-2} + C$

$\therefore \quad y = e^{-2} + Ce^{-x}$

As y is continuous

$\therefore \quad \lim\limits_{x\to 2} \dfrac{x+1}{e^x} = \lim\limits_{x\to 2}(e^{-2} + Ce^{-x})$

$\therefore \quad 3e^{-2} = e^{-2} + Ce^{-2} \Rightarrow C = 2$

$\therefore \quad$ for $x > 2$

$y = e^{-2} + 2e^{-x}$

$\Rightarrow \quad y(3) = 2e^{-3} + e^{-2} = e^{-2}(2e^{-1} + 1)$

$\Rightarrow \quad y' = -2e^{-x}$

$\Rightarrow \quad y'(3) = -2e^{-3}$

11. (a, b, c)

$$f(x) = \int_0^x \{f(t)\cos t - \cos(t-x)\}dx$$

$$= \int_0^x f(t)\cos t\,dt - \int_0^x \cos(-t)\,dt$$

$$\left[\text{using } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$\therefore \quad f(x) = \int_0^x f(t)\cos t\,dt - \sin x$

Differentiate both sides w.r.t. x,

$f'(x) = f(x)\cos x - \cos x$

or $\dfrac{dy}{dx} - y\cos x = -\cos x$

$\therefore \quad$ I.F. $= e^{-\int \cos x\,dx} = e^{-\sin x}$

Hence, solution is $y\,e^{-\sin x} = -\int e^{-\sin x}\cos x\,dx$

or $y\,e^{-\sin x} = C + e^{-\sin x}$

or $y = C\,e^{\sin x} + 1$

At $x = 0$; $y = 0 \Rightarrow C = -1$

$\therefore \quad f(x) = 1 - e^{\sin x}$

$f_{\text{min.}} = 1 - e$ (when $x = \pi/2$)

and $f_{\text{max.}} = 1 - e^{-1}$ (when $x = -\pi/2$)

$f'(x) = -e^{\sin x}\cos x$

$\Rightarrow \quad f'(0) = -1$

Also $f''(x) = -[\cos^2 x\, e^{\sin x} - e^{\sin x}\cdot \sin x]$

$f''\left(\dfrac{\pi}{2}\right) = e$

12. (a, b, c, d)

$y(2xy + e^x)dx - e^x dy = 0$

$\therefore \quad y(2xy + e^x) = e^x \dfrac{dy}{dx}$

$\therefore \quad \dfrac{dy}{dx} - y = \dfrac{2xy^2}{e^x}$

$\therefore \quad \dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{y} = \dfrac{2x}{e^x}$

Put $\dfrac{1}{y} = v \Rightarrow \dfrac{1}{y^2}\dfrac{dy}{dx} = -\dfrac{dv}{dx}$

$\therefore \quad -\dfrac{dv}{dx} - v = \dfrac{2x}{e^x}$

$\therefore \quad \dfrac{dv}{dx} + v = -\dfrac{2x}{e^x}$

I.F. $= e^{\int 1dx} = e^x$

Solution $v.\,e^x = \int \dfrac{-2x}{e^x}e^x dx + c$

or $\dfrac{e^x}{y} = -x^2 + c$

Curve passes through the point $(0, 1)$

$\Rightarrow \quad c = 1$

$\therefore \quad$ Curve is $y^{-1}e^x = 1 - x^2$

or $y = \dfrac{e^x}{1-x^2}$

Clearly $\lim\limits_{x\to\infty} f(x) = -\infty$ and $\lim\limits_{x\to-\infty} f(x) = 0$

Also $y' = \dfrac{e^x(1 - x^2 + 2x)}{(1-x^2)^2}$

$y' = 0 \Rightarrow x = 1 \pm \sqrt{2}$

From sign scheme, we can check that $x = 1 + \sqrt{2}$ is the point of local maxima and $x = 1 - \sqrt{2}$ is point of local minima.

DPP 10.3

Single Correct Answer Type

1. (c) Given differential equation can be written as

$$\frac{xdy + ydx}{(xy-1)^2} = \frac{dx}{x^2} \quad \Rightarrow \quad \frac{d(xy)}{(xy-1)^2} = \frac{dx}{x^2}$$

Integrating both sides

$$-\frac{1}{(xy-1)} = -\frac{1}{x} + c \Rightarrow \frac{1}{xy-1} = \frac{1}{x} + c$$

2. (d) $x\,dy + 2y^3\,dy = y\,dx$

$$\Rightarrow \quad 2y\,dy = \frac{y\,dx - x\,dy}{y^2} \quad \Rightarrow \quad 2y\,dy = d\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad x = y(c + y^2)$$

3. (c) $\dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} + \dfrac{x\,dy - y\,dx}{x^2} = 0$

$$\Rightarrow \quad \frac{1}{2}\frac{d(x^2+y^2)}{\sqrt{x^2+y^2}} + d\left(\frac{y}{x}\right) = 0$$

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$\Rightarrow \quad \sqrt{x^2+y^2}+\frac{y}{x}=c$

4. (a) $(3yx^2)dx - x^3dy = 2y^4dy$

$\Rightarrow \quad yd(x^3) - x^3dy = 2y^4dy$

$\Rightarrow \quad \frac{yd(x^3)-x^3dy}{y^2} = 2y^2dy$

$\Rightarrow \quad d\left(\frac{x^3}{y}\right) = 2y^2dy$

$\Rightarrow \quad \frac{x^3}{y} = \frac{2}{3}y^3 + c$

5. (c) $\frac{x\,dy - y\,dx}{y^2} = dy$

$\Rightarrow \quad -\int d\left(\frac{x}{y}\right) = \int dy$

$\Rightarrow \quad -\frac{x}{y} = y + c$

At $x = 1, y = 1 \quad \therefore \quad c = -2$

At $x = -3, \ \frac{3}{y} = y - 2$

$\Rightarrow \quad y^2 - 2y - 3 = 0$

$\Rightarrow \quad y = 3$

6. (b) The equation can be written as

$x^2\frac{x\,dy - y\,dx}{x^2} = x\sqrt{1-(y/x)^2}\,dx$

$\Rightarrow \quad \frac{d(y/x)}{\sqrt{1-(y/x)^2}} = \frac{dx}{x}$

$\Rightarrow \quad \sin^{-1} y/x = \log|x| + c$

Since $y(1) = 0$, $c = 0$.

Hence $y = x\sin(\log|x|)$

7. (c) The given equation can be written as

$y(1+x^{-1})dx + (x + \log x)dy + \sin y\,dx + x\cos y\,dy = 0$

$\Rightarrow \quad d(y(x + \log x)) + d(x\sin y) = 0$

$\Rightarrow \quad y(x + \log x) + x\sin y = C$

8. (b) $(x+1)\frac{dy}{dx} + 1 = \frac{e^x}{e^y}$

$\Rightarrow \quad e^y(x+1)\frac{dy}{dx} + e^y = e^x$

$\int \frac{d}{dx}(e^y(x+1))\,dx = \int e^x dx$

$\Rightarrow \quad e^y(x+1) = e^x + c$

9. (c) $\frac{y\,dx + x\,dy}{xy} = \left(\cot\frac{y}{x}\right)\left(\frac{x\,dy - y\,dx}{x^2}\right)$

$\Rightarrow \quad \frac{1}{xy}d(xy) = \cot\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$

$\Rightarrow \quad \log xy = \log\sin\left(\frac{y}{x}\right) + \log c$

$\Rightarrow \quad xy = c\sin\frac{y}{x}$

10. (c) $\sqrt{1-x^2y^2}\,dx = y\,dx + x\,dy$

$\Rightarrow \quad dx = \frac{d(xy)}{\sqrt{1-(xy)^2}}$

$\Rightarrow \quad x + c = \sin^{-1}(xy)$

$\Rightarrow \quad xy = \sin(x + c)$

11. (b) $y\,dx + x\,dy + xy^2dx - x^2y\,dy = 0$

$\Rightarrow \quad \frac{y\,dx + x\,dy}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0.$

On integrating, we get

$-\frac{1}{xy} + \log x - \log y = k \Rightarrow \log\frac{x}{y} = \frac{1}{xy} + k$

12. (a) $\frac{dy}{dx} = \frac{y^3}{e^{2x}+y^2}$

Dividing by e^{2x}

$\therefore \quad \frac{dy}{dx} = \frac{y^3e^{-2x}}{1+y^2e^{-2x}}$

$\therefore \quad dy + y^2e^{-2x}dy = y^3e^{-2x}dx$

$\therefore \quad \int 2\frac{dy}{y} + \int 2(ye^{-2x}dy - y^2e^{-2x}dx) = 0$

$\therefore \quad \int 2\frac{dy}{y} + \int d(e^{-2x}y^2) + c = 0$

$\therefore \quad 2\ln|y| + e^{-2x}y^2 = c$

DPP 10.4

Single Correct Answer Type

1. (c) According to the question

$\frac{dP}{dt} = 0.1P$, where P is population at time t.

$\therefore \quad \frac{dP}{P} = 0.1dt$

Integrating we get $\log_e P = 0.1t + c$

$\therefore$ Let at $t = 0$, $P = P_0$

$\therefore \quad \log_e P_0 = 0 + c$

$\therefore \quad \log_e P = 0.1t - \log_e P_0$

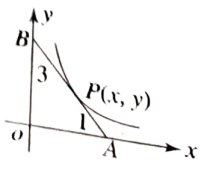
$\therefore \quad \log_e(P/P_0) = 0.1t$

For double population we have

$\therefore \quad \log_e(2P_0/P_0) = 0.1t$

$\therefore \quad t = 10\log_e 2$

2. (b)



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$\frac{3X}{x}+\frac{Y}{y}=4$ is equation of tangent at $P(x, y)$ having slope $-\frac{3y}{x}$

$\therefore \quad \frac{dy}{dx}=-\frac{3y}{x} \Rightarrow x\frac{dy}{dx}+3y=0$

$\therefore \quad \frac{dy}{y}=-3\frac{dx}{x}$

$\Rightarrow \quad \log y=-3\log x+\log c$

$\Rightarrow \quad y=\frac{c}{x^3}$ which passes through (1,1)

$\Rightarrow \quad c=1$

Equation of the curve is $y=\frac{1}{x^3}$

3. (c) Clearly PQ = Length of tangent

$\therefore \quad PQ=y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

Given $PQ=1$

$\therefore \quad y\sqrt{1+\left(\frac{dx}{dy}\right)^2}=1$

$\therefore \quad \left(\frac{dy}{dx}\right)^2=\frac{y^2}{1-y^2}$

$\therefore \quad \frac{dy}{dx}=\frac{-y}{\sqrt{1-y^2}}$ (as curve lies in first quadrant)

Putting $y=\sin\theta$

$\therefore \quad \cos\theta\frac{d\theta}{dx}=\frac{-\sin\theta}{\cos\theta}$

$\therefore \quad -\frac{\cos^2\theta}{\sin\theta}d\theta=dx$

$\therefore \quad \left(\sin\theta-\frac{1}{\sin\theta}\right)d\theta=dx$

$\therefore \quad -\cos\theta-\log_e\tan\frac{\theta}{2}=x+c$

$y=1$ when $x=0$

$\therefore \quad x=0,\ \theta=\pi/2 \ \therefore\ c=0$

$\therefore$ The curve in parametric form is

$$x=-\cos\theta-\log_e\tan\left(\frac{\theta}{2}\right),\ y=\sin\theta$$

4. (a) Equation of tangent at the point $R(x, f(x))$ is $Y-f(x)=f'(x)(X-x)$

Coordinates of point P are $(0, f(x)-xf'(x))$

The slope of the perpendicular line through 'P' is

$$\frac{f(x)-xf'(x)}{-1}=-\frac{1}{f'(x)}$$

$\Rightarrow \quad y\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2=1$ is differential equation.

5. (b) Differentiating, we have

$$a^{n-1}\frac{dy}{dx}=nx^{n-1}$$

$\Rightarrow \quad a^{n-1}=n\,x^{n-1}\frac{dx}{dy}$

Putting this value in the given equation, we have

$$nx^{n-1}\frac{dx}{dy}\,y=x^n$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we have $ny=-x\frac{dx}{dy}$

$\Rightarrow \quad ny\,dy+x\,dx=0$

$\Rightarrow \quad ny^2+x^2=\text{const.}$

Which is the required family of orthogonal trajectories.

Multiple Correct Answers Type

6. (a, d)

If (x, y) is any point on the curve, the sub tangent at (x, y)

$=y\frac{dx}{dy}$

$\therefore \quad y\frac{dx}{dy}=nx$ (given)

or $\quad n\frac{dy}{y}=\frac{dx}{x}$

Integrating $n\log y=\log x+\log c$

or $\quad \log y^n=\log cx$

or $\quad y^n=cx$ (i)

which is the required equation of the family of curves.

Putting $x=2, y=3$ in (i), we have $3^n=2c$ or $c=\frac{3^n}{2}$

Putting this value of c in (i),

$$y^n=\frac{3^n}{2}x$$

or $\quad 2y^n=3^nx$ (ii)

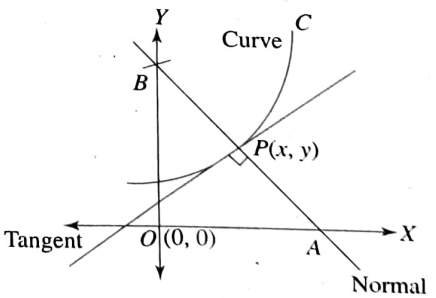
Which is the particular curve passing through the point (2, 3)

Putting $n=1$ in (ii), we have $2y=3x$

Which is a straight line

Putting $n=2$ in (ii), we have $2y^2=9x$.

7. (a, d)



The equation of normal at $P(x, y)$ is $(Y-y)=\frac{-1}{\frac{dy}{dx}}(X-x)$

$\therefore \quad A\left(x+y\frac{dy}{dx}, 0\right)$ and $\left(0, y+\frac{x}{\frac{dy}{dx}}\right)$

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Now $\dfrac{1\left(x+y\dfrac{dy}{dx}\right)+2(0)}{1+2}=x \Rightarrow x+y\dfrac{dy}{dx}=3x$

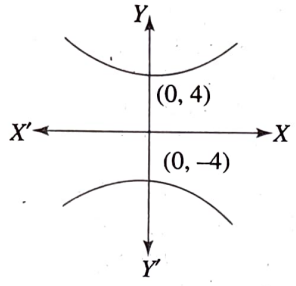
$\Rightarrow \quad y\dfrac{dy}{dx}=2x \qquad (1)$

$\Rightarrow \quad \dfrac{y^2}{2}=x^2+C$

Also (0, 4) satisfy it, so $C = 8$.

$\therefore \quad y^2 = 2x^2 + 16$ (equation of curve)

Which represent a hyperbola



Also $\left.\dfrac{dy}{dx}\right]_{(4,4\sqrt{3})}=\dfrac{2(4)}{4\sqrt{3}}=\dfrac{2}{\sqrt{3}}$

8. (b, c, d)

Slope of normal $\tan\theta = -\dfrac{dx}{dy}$

$\therefore$ The given equation becomes at a general point (x, y) as

$$y\left(x^2\left(-\frac{dx}{dy}\right)+\frac{dy}{dx}\right)=x(y^2+1)$$

$$\Rightarrow \quad -yx^2+y\left(\frac{dy}{dx}\right)^2=\frac{dy}{dx}\cdot x(y^2+1)$$

$$\Rightarrow \quad y\left(\frac{dy}{dx}\right)^2-x(y^2+1)\frac{dy}{dx}-yx^2=0$$

$$\Rightarrow \quad yy'^2-xy^2y'-xy'-yx^2=0$$

$$\Rightarrow \quad yy'(y'-xy)-x(y'-xy)=0$$

$$\therefore \quad \frac{dy}{dx}=xy \text{ or } \frac{dy}{dx}=\frac{x}{y}$$

$$\therefore \quad \log y=\frac{x^2}{2}+c \text{ (or) } x^2-y^2=c$$

$$\therefore \quad y=ke^{\frac{x^2}{2}} \text{ (or) } \log y^2=x^2-\log k$$

Also $y = k \cdot e^{\frac{x^2}{2}} \Rightarrow \log ky^2 = x^2$

Comprehension Type

9. (b) Equation of Tangent at $P(x_1, y_1)$ of $y = f(x)$

$$y-y_1=\frac{dy}{dx}(x-x_1)$$

This tangent cuts the x-axis, where

$$x_2=x_1-\frac{y_1}{\left(\dfrac{dy}{dx}\right)}$$

$\because \quad x_1, x_2, x_3, \ldots x_n$ are in AP

$$x_2-x_1=-\frac{y_1}{\dfrac{dy}{dx}}=\log_2 e$$

$$-y=(\log_2 e)\frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{y}\log_2 e=-dx$$

Integrating both sides

$$\log y = -x \log_e 2 + \log c$$

$$\therefore \quad y = c\,2^{-x}$$

$\because y = f(x)$ passes through (0, 2).

$$2 = ce^0 \Rightarrow c = 2$$

$$\therefore \quad y = 2^{1-x}$$

$$\therefore \quad y(5)=\frac{1}{16}$$

10. (b) Again $x_2=x_1-\dfrac{y_1}{\left(\dfrac{dy}{dx}\right)}$

$\because \quad x_1, x_2, x_3, \ldots, x_n$ are in G.P.

$$\frac{x_2}{x_1}=1-\frac{y_1}{x_1\left(\dfrac{dy}{dx}\right)}$$

$$\Rightarrow \quad 1-\frac{y}{x}\frac{dx}{dy}=2$$

$$\Rightarrow \quad -\frac{y}{x}\frac{dx}{dy}=1$$

$$\Rightarrow \quad \frac{dx}{x}=-\frac{dy}{y}$$

Integrating,

$$\log x = -\log y + \log c$$

$$\Rightarrow \quad xy = c$$

Curve passing through (1, 2) $\Rightarrow \quad c = 2$

$\therefore \quad$ Curve is $xy = 2$.

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